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### Price rigidities and rationing

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**PRICE RIGIDITIES AND RATIONING**

by G. van der Laan and A.J.J. Talman

March, 1989

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# PRICE RIGIDITIES AND RATIONING <sup>1)</sup>

by

G. van der Laan <sup>2)</sup> and A.J.J. Talman <sup>3)</sup>

## Abstract

In this paper economic models are discussed and introduced that consider situations in which price restrictions and regulations prevent prices from adjusting according to the law of supply and demand. In most of these models markets are cleared through the adjustment of quantities, i.e., by rationing the excess supply or the excess demand. In equilibrium there is only rationing on at most one side of each market, while at least one market is not rationed at all. In particular there is such an equilibrium for which there is no demand rationing at all. In order to guarantee that at such a supply constrained equilibrium an a priori chosen commodity, for example money, is not rationed, we have to allow for price flexibility with respect to the price of this commodity. Supply rationing or unemployment has a serious impact on the income of the rationed agents. To deal with this problem we also consider models with unemployment compensations.

March 1989.

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## PRICE RIGIDITIES AND RATIONING

G. van der Laan and A.J.J. Talman

### 1. Introduction

Perfect competition is a basic assumption in classical economic theory. This implies that all agents in the economy are price takers and therefore express their demands and supplies of the commodities given the prevailing prices. A market is in equilibrium if total demand is equal to total supply. Trade will only take place at equilibrium prices. It is assumed that there are no restrictions on the prices and that prices adjust infinitely fast. Walras considered the problem of the existence of general equilibrium, i.e., a system of prices at which all markets clear simultaneously. Under rather general assumptions the existence of such a Walrasian price system has been proved in the fifties by Debreu and others.

Unemployment and excess supply on commodity markets are serious problems in many countries. Almost all semi-annual meetings of the government leaders of the European common market countries are mainly devoted to tackle problems like butter mountains, wine pools, milk lakes, grain warehouses, olive pyramids, dung-hills, and an unemployment army. In these cases price restrictions and regulations prevent prices from adjusting according to the law of supply and demand. So, prices can not reach their Walrasian equilibrium values; nevertheless, trade takes place. In fact, markets are cleared through the adjustment of quantities instead of prices, e.g., by imposing quotas.

In the mid seventies Drèze (1975) and Benassy (1975) independently developed models for equilibrium under price rigidities, such as price controls, minimum wages or price indexation. Both authors introduced the concept of quantity rationing in general equilibrium models under restrictions on the prices. In this approach an agent chooses that commodity bundle which is most preferred by him, subject to both his budget constraint and to quantity constraints on net trade. The quantity rationing may affect either supply or demand of a commodity, but it never affects both simultaneously. It is further assumed that no quantity restrictions are allowed unless price rigidities are binding.

The models of Drèze and Benassy differ in the way they describe the behaviour of the individual agents under quantity rationing. In Drèze's model, the agents express their rationed demands and supplies, i.e., on each market the agents' demands and supplies satisfy the imposed rationing scheme. This kind of behaviour has also been studied by Younès (1975), Grandmont and Laroque (1976), Hahn (1978), Laroque (1978), van der Laan (1980), and Kurz (1982). In the model of Benassy the agents express their effective demands and supplies. This notion goes back to Clower (1965) and Barro and Grossman (1971) and reflects the agent's demand or supply for a commodity when he takes into account only the rationing on the other commodities. For example, if a consumer is constrained in the demand for milk, he compensates his desire for milk through a higher demand for wine, irrespective of the rationing on the wine market. On the other hand, his effective demand for milk will not take into account the rationing on milk and typically exceeds the level of rationing, due to rationing on wine and other substitutes. Whereas for Drèze's model the expressed and realized quantities of trade are in equilibrium equal to each other, in the model of Benassy trade is realized through assignments induced by the effective demands and supplies. These assignments yield new perceived rationing schemes. Equilibrium is defined as a state in which agents have no incentive to adjust their effective demands or supplies according to these new schemes.

Under price rigidities, Drèze and Benassy proved the existence of an equilibrium such that at least one a priori chosen numeraire commodity is not rationed at all, while for the other commodities there is rationing on the demand or the supply side, but not on both sides of the same market simultaneously. If money is one of the commodities, this commodity can be chosen to be the numeraire, implying that there exists an equilibrium with no rationing on money.

In practice, rationing of supplies seems to occur much more frequently than rationing of demands. Moreover, both van der Laan (1980) and Kurz (1982) argue that in practice it is difficult to implement rationing of demand. These observations lead these authors to prove the existence of a so-called supply-constrained or unemployment equilibrium. In such an equilibrium only supplies are rationed whereas at least one commodity is not rationed at all. However, which commodities are not rationed is not known in advance, so that it can not be assumed that there is an unemployment equilibrium in which the supply of an a priori chosen commodity is unrationed. Not surprisingly, more price flexibility is needed to assure the existence of an unemployment equilibrium in which an a priori given commodity is not rationed. This topic has been attacked by van der Laan (1984), Dehez and Drèze (1984), and Weddepohl (forthcoming), by introducing flexible money prices.

Rationing of the supplies has a serious impact on the income of the rationed agents. Therefore van der Laan (1980, 1981) considered models with unemployment compensations for unemployed people.

This paper has been organized as follows. In the next section we give the basic assumptions about the Walrasian model of an exchange economy. In Section 3 we introduce price rigidities and define constrained equilibria. Section 4 deals with effective demand and treats the model of Benassy. This model yields the so-called neo-Keynesian or K-equilibrium. In Section 5 the different types of K-equilibria are discussed. Also we give a characterization of the types of equilibria for the well-known Malinvaud model. In Section 6 we discuss the problem of manipulable rationing schemes. The existence of constrained equilibria will be proved in Section 7. After a discussion about supply-constrained and demand-constrained equilibria in Section 8, the existence of supply-constrained equilibria without rationing on the money commodity is proved in Section 9. In Section 10 we motivate the occurrence of supply-constrained equilibria. In the following two sections several models with unemployment compensations are considered, in Section 11 some models without money and in Section 12 a model for a monetary economy. In the latter model the policy of compensations results in inflationary or deflationary impulses.

## 2. Preliminaries

We consider an exchange economy with  $n+1$  commodities, indexed  $j = 0, 1, \dots, n$ , and  $m$  agents, indexed  $i = 1, \dots, m$ . For simplicity, we assume that each agent represents a consumer (or household), who maximizes his utility under the budget constraint and quantity constraints. Consumer  $i$ ,  $i = 1, \dots, m$ , is characterized by a consumption set  $X^i$ , a utility function  $u^i$  on  $X^i$  representing his preferences, and a vector of initial endowments  $w^i \in \mathbb{R}^{n+1}$ . Let  $\Omega = \{x \in \mathbb{R}^{n+1} \mid x_j \geq 0, j = 0, \dots, n\}$  denote the nonnegative orthant of  $\mathbb{R}^{n+1}$ . For all  $i$ , we make the following assumptions, where  $w = \sum_i w^i$  is the vector of total initial endowments.

$A_1$ . The consumption set  $X^i$  is a compact <sup>1)</sup>, convex subset of  $\Omega$ , containing the set  $\{x \in \Omega \mid x_j \leq w_j, j = 0, \dots, n\}$ .

<sup>1)</sup> It is sufficient to assume that the sets  $X^i$  are closed. However to simplify the proofs we assume that the sets are compact, i.e., closed and bounded.

A<sub>2</sub>. The utility function is a strictly quasi-concave <sup>2)</sup> continuous function from  $X^i$  to  $R$ , satisfying monotonicity, i.e., for all  $x, y \in X^i$ ,  $x_j \geq y_j$  for all  $j$  and  $x_j > y_j$  for at least one  $j$  implies  $u^i(x) > u^i(y)$ .

A<sub>3</sub>. For each commodity  $j$ ,  $w_j^i > 0$ .

Given  $p$  in  $\Omega \setminus \{0\}$ , let  $d^i(p) \in X^i$  be a consumption bundle satisfying consumer  $i$ 's budget constraint  $p^T x \leq p^T w^i$ , such that no other  $x \in X^i$  satisfying the budget constraint is preferred to  $d^i(p)$ . Under the assumptions A<sub>1</sub>, A<sub>2</sub>, and A<sub>3</sub> such a consumption bundle exists and is preferred to all other  $x \in X^i$  satisfying the budget constraint, which implies that  $d^i(p)$  is unique. Moreover, the assumptions imply that consumer  $i$ 's demand function  $d^i: \Omega \setminus \{0\} \rightarrow X^i$  is continuous. Let  $z(p)$  denote the total excess demand at price  $p$ , i.e.,

$$z(p) = \sum_i (d^i(p) - w^i).$$

Then the function  $z$  is a continuous function from  $\Omega \setminus \{0\}$  to  $R^{n+1}$  satisfying for all  $p \in \Omega \setminus \{0\}$ ,

- i)  $p^T z(p) = 0$  (Walras' law, i.e., the total value of the excess demands is equal to zero),
- ii)  $z(\lambda p) = z(p)$  for all  $\lambda > 0$  (homogeneity of degree zero),
- iii)  $z_j(p) \geq 0$  if  $p_j = 0$  (desirability).

A price vector  $p^*$  is called a Walrasian equilibrium price vector if  $z(p^*) = 0$ , i.e., if  $p^*$  is a zero point of  $z$ . For the proof of the existence of such a price vector we refer to Debreu (1959) and others.

### 3. Constrained equilibrium

Drèze (1975) considered the problem of restrictions on the prices. In his model the commodity indexed by  $j = 0$  serves as the numeraire commodity. Because of the homogeneity property its price can be set equal to one without loss of generality. The prices of the other commodities are restricted from below and above by constants  $\underline{p}_j$  and  $\bar{p}_j$  for commodity  $j$ ,  $j = 1, \dots, n$ . In a later section we will discuss more general price restrictions. Assuming that for all  $j \neq 0$ ,  $0 < \underline{p}_j \leq \bar{p}_j < \infty$ , the nonempty set  $P_0$  of admissible prices becomes

<sup>2)</sup> It is sufficient to assume quasi-concavity of the utility functions. However, assuming strict quasi-concavity implies that the demand functions are continuous functions instead of upper semi-continuous multifunctions.



$$P_0 = \{p \in \Omega \mid p_0 = 1, p_j \leq \bar{p}_j \text{ for all } j \neq 0\}.$$

Because of the restrictions on the prices, an equilibrium price vector  $p^*$  does not need to be an element of  $P_0$ . Drèze (1975) defined an equilibrium concept involving quantity constraints on the individual excess supplies and excess demands. For a price vector  $p \in P_0$ , a vector  $l^i \in -\Omega$  of constraints on the net supplies of consumer  $i$ , and a vector  $L^i \in \Omega$  of constraints on the net demands of  $i$ , the constrained budget set of consumer  $i$  becomes

$$B^i(p, l^i, L^i) = \{x \in X^i \mid p^T x \leq p^T w^i, l^i \leq x - w^i \leq L^i\}.$$

The constrained demand of consumer  $i$ , denoted  $d^i(p, l^i, L^i)$ , is defined as the element in the constrained budget set of  $i$  which maximizes  $i$ 's utility. Because of the strict quasi-concavity of  $u^i$  this element is uniquely determined. We say that agent  $i$  is rationed on the demand (supply) of commodity  $j$  if  $y^i$  is preferred to  $x^i = d^i(p, l^i, L^i)$  where  $y^i$  maximizes  $i$ 's utility in the budget set of  $i$  when  $L^i_j$  is increased ( $l^i_j$  decreased respectively) with an arbitrarily small amount. The strict quasi-concavity of the utility function implies that a consumer is not rationed on the demand (or supply) of commodity  $j$  if  $x^i_j - w^i_j < L^i_j$  (or  $x^i_j - w^i_j > l^i_j$ ).

**Definition 3.1.** A constrained equilibrium is a set of consumptions  $x^i \in X^i$ ,  $i = 1, \dots, m$ , a set of supply rationing schemes  $l^i \in -\Omega$  and demand rationing schemes  $L^i \in \Omega$ ,  $i = 1, \dots, m$ , and a price vector  $p \in P_0$  such that

- a) for all  $i$ ,  $x^i = d^i(p, l^i, L^i)$ ,
- b)  $\sum_i x^i = w$ ,
- c) for all  $j$ ,  $x^i_j - w^i_j = L^i_j$  for some  $i$  implies  $x^h_j - w^h_j > l^h_j$  for all  $h$ , and  $x^i_j - w^i_j = l^i_j$  for some  $i$  implies  $x^h_j - w^h_j < L^h_j$  for all  $h$ ,
- d) for all  $j$ ,  $p_j < \bar{p}_j$  implies  $L^i_j > x^i_j - w^i_j$  for all  $i$ , and  $p_j > \underline{p}_j$  implies  $l^i_j < x^i_j - w^i_j$  for all  $i$ .

Conditions a) and b) require that the consumption of each agent equals his constrained demand and that the total consumption equals the total initial endowment. Condition c) implies that at a constrained equilibrium not both sides of a market are rationed simultaneously. We say that a market is frictionless when there is rationing on at most one side of the market. So, condition c) requires that in equilibrium each market must be frictionless. Finally, condition d) guarantees that in equilibrium there is no demand rationing if the price is not on its upper bound and that there is no supply rationing if the price is not on its lower bound.

There are two trivial constrained equilibria:  $x^i = w^i$  for all  $i$ ,  $l^i = 0$  for all  $i$ ,  $L^i_j > 0$  for all  $i$  and  $j$ , and  $p = \underline{p}$ ; and  $x^i = w^i$  for all  $i$ ,  $l^i_j < 0$  for all  $i$  and  $j$ ,  $L^i = 0$  for all  $i$ , and  $p = \bar{p}$ . In the first case there is complete rationing on all excess supplies, so that for all  $i$   $w^i$  is the unique element of  $i$ 's constrained budget set of  $i$  and therefore satisfies condition a). In the second case there is complete rationing on all excess demands. Then the budget set of  $i$  becomes

$$B^i(p, l^i, L^i) = \{x \in X^i \mid p^T x \leq p^T w^i, l^i \leq x - w^i \leq 0\},$$

so that  $d^i(p, l^i, L^i)$  equals  $w^i$  due to the monotonicity assumption. We conclude this section with two definitions.

**Definition 3.2.** A constrained equilibrium is a Drèze equilibrium if for all  $i$ ,  $l^i_0 = -\infty$  and  $L^i_0 = \infty$ .

**Definition 3.3.** A constrained equilibrium is a supply-constrained (or unemployment) equilibrium if for all  $i$ ,  $L^i_j = \infty$  for all  $j$  and  $l^i_j = -\infty$  for at least one  $j$ .

Observe that both the Drèze and the supply-constrained equilibrium are nontrivial equilibria. In the Drèze equilibrium there is no rationing on the numeraire commodity. In a supply-constrained equilibrium there is no rationing on the demand side, while at least one commodity is not constrained on the supply side. The existence of Drèze and supply-constrained equilibria will be proved in Section 7.

#### 4. Effective demands and K-equilibria

In the Drèze model it is assumed that an agent reveals his constrained demand  $d^i(p, l^i, L^i)$ . This constrained demand is the response of the agent to the message  $(p, l^i, L^i)$  called by an auctioneer. In this response the consumer takes into account his quantity constraints. Because the constrained demands are revealed, we have that in equilibrium there is no difference between the expressed demands and the realized transactions. All agents choose a consumption bundle out of their constrained budget set and the expressed demand satisfies therefore the rationing schemes. So, the constrained demands do not reveal binding constraints, i.e., a constrained agent does not reveal his desire to trade more. Now, suppose we have consumption bundles  $x^i \in X^i$ ,  $i = 1, \dots, m$ , a set of rationing schemes  $l^i \in -\Omega$  and  $L^i \in \Omega$ ,  $i = 1, \dots, m$ , and a price vector  $p \in P_0$  such that the conditions a), b) and d) of Definition 3.1 are satisfied, but condition c) is violated. In this case at least one market is not frictionless, i.e., on at least one market some agents are constrained in their excess demands and some others

in their excess supplies. Both the demand-constrained and supply-constrained agents can be made better off by trading more. However, there are no signals about this desire to trade more, because the agents express just their constrained demands. So, the economy can get stuck in this situation, violating condition c) of frictionless markets in equilibrium. To get out of such situations the agents have to reveal their desired trades instead of their constrained demands.

Following Clower (1965), Leijonhufvud (1968) and others, Benassy (1975) used the concept of effective demand. The effective demand of an agent for a commodity reveals an individual's offer to buy or to sell on the market, and not the actually realized constrained transaction. This effective demand is expressed separately on each market and does not take into account the rationing scheme perceived on that market. To reach equilibrium, each agent is assigned a transaction and perceives therefore new quantity constraints on his exchange. Due to these new constraints, each agent will express new effective demands, and so on. An equilibrium is reached when the new perceived constraints coincide with the previous ones and all markets are frictionless. This type of disequilibrium models in which the expressed demands may differ from the realized transactions has been studied by Barro and Grossman (1971), Malinvaud (1977), and in a general context by Benassy (1975, 1982).

In Benassy's analysis the price vector is completely fixed, i.e.,  $p_j = \bar{p}_j$  for all  $j$  and hence  $P_0$  contains only one element, denoted by  $p$ . The numeraire commodity  $j = 0$  is assumed to be money, which serves as the sole medium of exchange. Consumers derive utility from money as a store of value. It is assumed that money buys commodities and vice versa, but commodities do not buy commodities. So, there are  $n$  markets, and transactions between commodities and money take place on each market. The difference between an individual's terminal holding and initial holding of money is equal to the difference between the money values of the initial and terminal bundle of commodities. On each market rationing may occur. The reasoning above implies that there is only rationing on the demand or supply of the consumption goods and labour and not on money. So, in this section the rationing scheme of a consumer  $i$  will be a pair  $(l^i, L^i)$  with  $l^i \in -\Omega$ ,  $L^i \in \Omega$ , and  $-l^i_0 = L^i_0 = \infty$ .

When  $e^i = (e^i_1, \dots, e^i_n)^T$  is a demand vector of agent  $i$  for the non-money commodities, then, given the rationing scheme  $(l^i, L^i)$ , the resulting trade of commodity  $j$  for agent  $i$  will be

$$z^i_j(e^i, l^i, L^i) = \max(l^i_j, \min(e^i_j - w^i_j, L^i_j)), j = 1, \dots, n.$$

To be sure that the resulting assignments  $x^i_j(e^i, l^i, L^i) = w^i_j + z^i_j(e^i, l^i, L^i)$  of the non-numeraire commodities  $j \neq 0$  and the resulting terminal holding  $x^i_0(e^i, l^i, L^i) = w^i_0 - \sum_j p_j z^i_j(e^i, l^i, L^i)$  of the numeraire commodity are feasible, any demand vector  $e^i \in R^n$  of agent  $i$  must be restricted to the set

$$Z^i(l^i, L^i) = \{e^i \in R^n \mid x^i(e^i, l^i, L^i) \in X^i\},$$

with  $x^i(e^i, l^i, L^i)$  the  $(n+1)$ -vector with components  $x^i_j(e^i, l^i, L^i)$ ,  $j = 0, \dots, n$ . The demand vector expressed by agent  $i$  should be an element in  $Z^i(l^i, L^i)$  which maximizes the resulting utility  $u^i(x^i(e^i, l^i, L^i))$  over all  $e^i$  in  $Z^i(l^i, L^i)$ . Let  $E^i(l^i, L^i)$  be the set of all such elements in  $Z^i(l^i, L^i)$ . Clearly, the constrained demand vector  $c^i_j(l^i, L^i)$  where  $c^i_j(l^i, L^i) = d^i_j(p, l^i, L^i)$  for the non-numeraire commodities  $j = 1, \dots, n$  belongs to the set  $E^i(l^i, L^i)$  since the constrained demand  $d^i(p, l^i, L^i)$  maximizes  $u^i$  under the quantity constraints  $l^i$  and  $L^i$ . However, generally the set  $E^i(l^i, L^i)$  contains more than one element. In particular, we will show that one of these elements is the effective demand.

The effective demand on a market is defined as the utility maximizing demand for that commodity without taking into account the quantity constraints on that market (see Benassy (1982)). Formally, the effective demand  $e^i_j(l^i, L^i)$  of consumer  $i$  for commodity  $j \neq 0$  is the  $j$ -th component of the vector which solves the problem

$$\begin{aligned} &\max u^i(x) \text{ such that} \\ &x \in X^i, p^\top x \leq p^\top w^i, \text{ and } l^i_h \leq x_h - w^i_h \leq L^i_h, h \neq 0, j. \end{aligned} \quad (P^j_i)$$

In this constrained maximization problem the consumer takes into account all quantity constraints except the constraints on market  $j$ . Recall that there are no constraints for  $j = 0$ , so that  $h \neq 0, j$  might be replaced by  $h \neq j$ . Let  $\phi^{ij}(l^i, L^i)$  be the solution to problem  $(P^j_i)$ . Solving this problem for each  $j = 1, \dots, n$ , we obtain the effective demand vector  $e^i(l^i, L^i) \in R^n$  of consumer  $i$ ,  $i = 1, \dots, m$ , by taking for each commodity  $j$  the  $j$ -th component of the corresponding solution vector, i.e.

$$e^i_j(l^i, L^i) = \phi^{ij}_j(l^i, L^i), j = 1, \dots, n.$$

Clearly, in general the effective excess demand  $e^i_j(l^i, L^i) - w^i_j$  does not satisfy the rationings  $l^i_j$  and  $L^i_j$  on market  $j$ . However, under assumption  $A_2$  the effective demand vector belongs to  $E^i(l^i, L^i)$ . We remark that concavity of the utility function is not sufficient (see Grandmont (1977)). In the following we denote the trade



$z_j^i(e^i(l^i, L^i), l^i, L^i)$  on market  $j$  resulting from the effective demand  $e^i(l^i, L^i)$  by  $z_j^i(l^i, L^i)$ ,  $j = 1, \dots, n$ .

Theorem 4.1. Under  $A_2$ ,  $e^i(l^i, L^i) \in E^i(l^i, L^i)$  holds.

Proof: Following Benassy (1982, pages 188/9) it can be shown that under strict quasi-concavity of the utility function  $u^i$ , the trade  $z_j^i(l^i, L^i)$  on market  $j$  is equal to the constrained excess demand  $c_j^i(l^i, L^i) - w_j^i$ ,  $j = 1, \dots, n$ . Since  $c^i(l^i, L^i)$  belongs to  $E^i(l^i, L^i)$  this proves the theorem.

The set of rationing schemes  $(l^i, L^i)$ ,  $i = 1, \dots, m$ , is said to constitute an effective demand equilibrium if  $\sum_i z_j^i(l^i, L^i) = 0$ ,  $j = 1, \dots, n$ . From the fact that  $z_j^i(l^i, L^i)$  equals  $c_j^i(l^i, L^i) - w_j^i$  for all  $i$  and  $j$ , it follows that an effective demand equilibrium yields a constrained equilibrium only when the realized consumption  $x_j^i = z_j^i(l^i, L^i) + w_j^i$  satisfies condition c) of Definition 3.1 for all  $i$  and for all  $j \neq 0$ . This condition of frictionless markets possibly may not hold at an effective demand equilibrium. Observe that condition d) is redundant because  $P_0$  contains only  $p$  as the unique element. If, for some  $j$ ,  $\sum_i z_j^i(l^i, L^i) \neq 0$ , then transactions on market  $j$  can not be carried out and we do not have a constrained equilibrium.

To reach a constrained equilibrium, a set of assignment functions  $F^i: R^{nm} \rightarrow R^n$ ,  $i = 1, \dots, m$ , is introduced. These assignment functions are such that on each market the total assignments to the demanders is equal to the total assignments to the suppliers whereas on the short side of the market the agents realize their effective excess demands. More precisely, if  $\tilde{e} = (\tilde{e}^1, \dots, \tilde{e}^m) \in R^{nm}$  is a set of effective excess demand vectors given by  $\tilde{e}_j^i = e_j^i(l^i, L^i) - w_j^i$ ,  $j = 1, \dots, n$ ,  $i = 1, \dots, m$ , then for all  $j$  the assignments  $z_j^i = F_j^i(\tilde{e})$ ,  $i = 1, \dots, m$  satisfy the following conditions:

- 1)  $\sum_i z_j^i = 0$ ,
- 2) for all  $i$ ,  $0 \leq z_j^i \leq \tilde{e}_j^i$  if  $\tilde{e}_j^i \geq 0$ , and  $\tilde{e}_j^i \leq z_j^i \leq 0$  if  $\tilde{e}_j^i \leq 0$ ,
- 3) for all  $i$ ,  $z_j^i = \tilde{e}_j^i$  if  $\tilde{e}_j^i \geq 0$  and  $\sum_i \tilde{e}_j^i \leq 0$ , and  $z_j^i = \tilde{e}_j^i$  if  $\tilde{e}_j^i \leq 0$  and  $\sum_i \tilde{e}_j^i \geq 0$ .

Condition 1) says that the assignments can be realized, condition 2) that one can not force any agent to exchange more than he wants and that all agents remain at the same side of the markets after assignment, and condition 3) that agents on the short side of the market can realize their effective excess demands. The last condition will imply frictionless markets, since on at most one side of the market agents can not realize their effective demands.

Given a set of effective excess demands the functions  $F^i$ ,  $i = 1, \dots, m$ , determine realizable transactions of each consumer for the non-money commodities. The resulting changes in money holdings follow immediately from the fact that commodities are exchanged against money, i.e.,  $z_0^i = -\sum_j p_j z_j^i$ ,  $i = 1, \dots, m$ .

Although the assignments are realizable, this does not imply that transactions will in fact take place. Through the assignments each individual  $i$  may perceive on market  $j$  a difference between his expected transaction  $z_j^i(l^i, L^i)$  and his assignment  $z_j^i$ . Based on this observation the individual wants to adjust his perceived rationing scheme. Since for all  $j$ , both  $z_j^i(l^i, L^i)$  and  $z_j^i$  are determined by the set of effective excess demand vectors, we may assume that this new scheme, say  $(k^i, K^i)$ , is a function of the effective excess demands only. More precisely, we assume that for all  $i$ ,  $i = 1, \dots, m$ ,

$$k^i = g^i(\tilde{e}) \text{ and } K^i = G^i(\tilde{e}),$$

with  $g^i: R^{nm} \rightarrow R^{n+1}$  and  $G^i: R^{nm} \rightarrow R^{n+1}$  functions satisfying  $g_0^i(\tilde{e}) = -\infty$  and  $G_0^i(\tilde{e}) = \infty$ , and for all  $j \neq 0$

$$\begin{aligned} 4) & g_j^i(\tilde{e}) < \min(0, z_j^i) \text{ and } G_j^i(\tilde{e}) > \max(0, z_j^i) \text{ if } z_j^i = \tilde{e}_j^i, \\ 5) & g_j^i(\tilde{e}) = z_j^i \text{ if } z_j^i > \tilde{e}_j^i \text{ and } G_j^i(\tilde{e}) = z_j^i \text{ if } z_j^i < \tilde{e}_j^i. \end{aligned}$$

These conditions say that the new rationing scheme allows for the assignments to be carried out and that the new rationing equals the assignment if and only if the individual is on the long side of the market and the assignment constraints his effective excess demand. These adjustments of the perceived rationing schemes helps us to escape from getting stuck in an equilibrium with non-frictionless markets.

Perceiving the new rationing scheme the individuals want to revise their effective demands. The effective demand of  $i$  will not change if the new rationing scheme  $(k^i, K^i)$  for  $i$  coincides with the old rationing scheme  $(l^i, L^i)$ . If this holds for all agents  $i$  the assignments yield a neo-Keynesian or K-equilibrium (see Benassy (1975, 1982)).

**Definition 4.2.** A K-equilibrium with respect to a price system  $p$  is a set of assignments  $z^i \in R^n$ , and effective excess demand vectors  $\tilde{e}^i \in R^n$ ,  $i = 1, \dots, m$ , and a set of rationing schemes  $l^i \in -\Omega$  and  $L^i \in \Omega$ , with  $l_0^i = -\infty$  and  $L_0^i = \infty$ ,  $i = 1, \dots, m$ , such that for all  $i$  and for  $j = 1, \dots, n$

$$\begin{aligned} (a) & \tilde{e}_j^i = e_j^i(l^i, L^i) - w_j^i, \\ (b) & z_j^i = F_j^i(\tilde{e}), \end{aligned}$$

$$(c) \, l_j^i = g_j^i(\tilde{e}) \text{ and } L_j^i = G_j^i(\tilde{e}).$$

Observe that at a K-equilibrium the expressed effective excess demands need not to be equal to the realized transactions after the assignments. If so, i.e., if  $z_j^i \neq \tilde{e}_j^i$ , then we obtain from condition 5) that  $g_j^i(\tilde{e}) = z_j^i$  if  $\tilde{e}_j^i < z_j^i$  and  $G_j^i(\tilde{e}) = z_j^i$  if  $\tilde{e}_j^i > z_j^i$ , and hence it follows from condition c) of Definition 4.2 that  $z_j^i = l_j^i$  if  $\tilde{e}_j^i < z_j^i$  and  $z_j^i = L_j^i$  if  $\tilde{e}_j^i > z_j^i$ . Moreover, if  $z_j^i = \tilde{e}_j^i$  it follows from condition 4) and from condition c) of Definition 4.2 that  $l_j^i \leq \tilde{e}_j^i = z_j^i \leq L_j^i$ . So, in equilibrium we have that for all  $i$  and  $j$

$$z_j^i = \max(l_j^i, \min(\tilde{e}_j^i, L_j^i)) = z_j^i(l^i, L^i).$$

Hence, for all  $j$  the assignment  $z_j^i$  is equal to  $z_j^i(l^i, L^i)$ . Since Theorem 4.1 says that  $e^i(l^i, L^i) \in E^i(l^i, L^i)$ , it follows that in a K-equilibrium for each consumer the vector of realized consumptions maximizes his utility in the constrained budget set and is equal to the constrained demand vector. Moreover condition 3) guarantees that agents on the short side can realize their effective excess demands. Together with condition 4) this implies that at a K-equilibrium all markets are frictionless. This gives us the next result.

**Corollary 4.3.** Let the set of transactions  $z^i$ , effective excess demand vectors  $\tilde{e}^i$ , and rationing schemes  $(l^i, L^i)$ ,  $i = 1, \dots, m$ , be a K-equilibrium with respect to  $p$ . Then with  $p$  as the vector of fixed prices, the set of consumptions  $x^i$  defined by

$$x_j^i = z_j^i + w_j^i, \quad j = 1, \dots, n \text{ and } x_0^i = w_0^i - \sum_j p_j z_j^i, \quad i = 1, \dots, m$$

and the set of rationing schemes  $(l^i, L^i)$ ,  $i = 1, \dots, m$ , constitute a constrained equilibrium with no quantity constraints on the numeraire commodity.

The property of frictionless markets implies that a K-equilibrium is efficient market by market. That means that for a set of K-equilibrium transactions  $z^i$  with rationing schemes  $(l^i, L^i)$ ,  $i = 1, \dots, m$ , there is no other feasible set of transactions  $y^i$ ,  $i = 1, \dots, m$ , yielding a higher utility for all agents, such that  $\sum_i y^i = 0$  and there exists a  $j \neq 0$  such that for all  $i$  and for all  $h \neq j$ ,

$$l_h^i \leq y_h^i \leq L_h^i.$$

So, it is not possible to make all agents better off by a more efficient allocation of just one of the commodities <sup>3)</sup>. The market by market efficiency follows from the fact that for all  $i$ ,  $x^i$  yields utility maximization under  $l_h^i < x_h^i - w_h^i \leq L_h^i$  for all  $h$ , i.e., the realized consumption  $x^i$  is equal to the constrained demand of  $i$ . Indeed, when each agent  $i$  would be better off under a  $y^i$  yielding, for some  $j \neq 0$ , maximal utility under  $l_h^i \leq y_h^i \leq L_h^i$  for all  $h \neq j$ , we must have that at the K-equilibrium all agents are constrained on market  $j$  while  $\sum_i y_j^i = 0$ . However, that contradicts the fact that there is only rationing on at most one side of the market.

Below we prove the existence of a K-equilibrium and hence the existence of a constrained equilibrium. Observe that this proof needs the specification of the functions  $g_j^i$  and  $G_j^i$  for all  $i$  and  $j$ , see also the end of Section 6.

**Theorem 4.4.** If for all  $i$  the functions  $g^i$  and  $G^i$  are continuous, then there exists a K-equilibrium.

**Proof:** For given rationing scheme  $(l^i, L^i)$  let  $x^i(j)$  maximize  $i$ 's utility without taking into account the constraints on market  $j$ . Since  $x^i(j) \in X^i$  we have that for all  $j$ ,  $x_j^i(j) \geq 0$ , while the budget constraint implies that  $x_j^i(j) - w_j^i \leq b_j$  with  $b_j = \max_i \sum_{h \neq j} p_h w_h^i / p_j$ . Hence for all  $i$  and  $j$ , the effective excess demand vectors  $\tilde{e}^i$  with  $\tilde{e}_j^i = x_j^i(j) - w_j^i$  satisfy  $-w_j^i \leq \tilde{e}_j^i \leq b_j$ . Let  $W$  be the subset of  $R^n$  defined by

$$W = \{y \in R^n \mid -w_j \leq y_j \leq b_j, j = 1, \dots, n\},$$

and let  $W^m = W \times W \times \dots \times W$  be the  $mn$ -dimensional cross product of  $m$  sets  $W$ . Furthermore, for  $y = (y^1, \dots, y^m) \in W^m$ , let  $h^i(y) \in R^n$  be defined by

$$h_j^i(y) = e_j^i(g^i(y), G^i(y)) - w_j^i, j = 1, \dots, n, i = 1, \dots, m.$$

Clearly,  $h^i(y) \in W$ . Since  $g^i$  and  $G^i$  are continuous functions, the strict quasi-concavity of the utility functions implies that  $h = (h^1, \dots, h^m)$  is a continuous function from  $W^m$  into itself. According to Brouwer's fixed point theorem there exists a  $y^*$  in  $W^m$  such that  $h^i(y^*) = y^{*i}$ ,  $i = 1, \dots, m$ . Clearly, the effective demand vectors  $y^{*i} \in R^n$ ,  $i = 1, \dots, m$ , constitute a K-equilibrium with transactions  $F^i(y^*)$  and rationing schemes  $(g^i(y^*), G^i(y^*))$ ,  $i = 1, \dots, m$ .

<sup>3)</sup> This does not exclude that there is a chain of traders  $(i_1, \dots, i_k)$  and a chain of commodities  $(j_1, \dots, j_k)$ , such that for all  $h = 1, \dots, k$ , trader  $i_h$  is constrained in the demand of good  $j_h$  and not constrained in the supply of good  $j_{h+1}$ , with  $h+1 = 1$  if  $h = k$ . In such a case a Pareto improvement is possible by weakening for all  $h$  the constraint of consumer  $i_h$  on the demand for good  $j_h$ .



A K-equilibrium is illustrated in Figure 4.1 for two consumers, A and B, and two non-money commodities. In this figure the origin denotes the no trade situation. On the horizontal axis we have the variables  $z^A_1$  to the right and  $z^B_1$  to the left, and on the vertical axis the variables  $z^A_2$  downwards and  $z^B_2$  upwards. Consequently, each point  $z$  denotes a feasible trade with  $z^A = (z_1, -z_2)$  and  $z^B = (-z_1, z_2)$ . If the transaction is carried out consumer  $i$ 's utility equals  $u^i(x^i)$  with  $x^i_j = w^i_j + z^i_j$ ,  $j=1,2$ , and  $x^i_0 = w^i_0 - \sum_j p_j z^i_j$ ,  $i = A, B$ . So, for both  $i$ , the indifference curves  $I^i$  of  $i$  are curves around the point  $m^i$  reflecting the unconstrained optimal excess demand. At the optimal points  $m^A$  and  $m^B$  the excess demand of consumer A for commodity 1 exceeds the excess supply of consumer B for that commodity, while for commodity 2 the excess demand of consumer B exceeds the excess supply of consumer A. This leads to demand rationing on both markets. Let  $L^A_1 < m^A_1$  be the demand rationing for A on the first market. Then, the effective excess demand for commodity 1 equals  $m^A_1$ , because A does not take into account this rationing in determining his effective demand for commodity 1. However, the effective excess demand for commodity 2 becomes equal to the second (negative) component of the point which maximizes  $u^A(x^A)$  under  $z^A_1 \leq L^A_1$ , i.e., the effective excess demand for commodity 2 becomes equal to the second component of the point where an indifference curve of A is tangent to the line  $z^A_1 = L^A_1$ . Let  $m^A(L^A_1)$  be this point, then  $(m^A_1, m^A_2(L^A_1))$  is the effective excess demand of consumer A given the rationing  $L^A_1$  on the demand of commodity 1. Similarly, with  $m^B(L^B_2)$  the tangent point on the indifference curve of B tangent to the line  $z^B_2 = L^B_2$ ,  $(m^B_1(L^B_2), m^B_2)$  is the effective excess demand of consumer B given the rationing  $L^B_2 < m^B_2$  on the demand of commodity 2. Given these effective demands, the assignments become  $(-m^B_1(L^B_2), m^A_2(L^A_1))$  for consumer A and  $(m^B_1(L^B_2), -m^A_2(L^A_1))$  for consumer B. An equilibrium is obtained if  $L^A_1 = -m^B_1(L^B_2)$  and  $L^B_2 = -m^A_2(L^A_1)$ , i.e., at the point K where the offer curves  $z^A = m^A(L^A_1)$ ,  $L^A_1 \geq 0$ , and  $z^B = m^B(L^B_2)$ ,  $L^B_2 \geq 0$ , intersect each other.

Insert Figure 4.1.

It should be observed that both agents prefer the point H above K. This shows that generally a K-equilibrium is not Pareto optimal nor even optimal with respect to the price system  $p$ . That means, given  $p$ , there exists feasible transactions  $y^i$ , satisfying the budget constraint  $x^i_0 = w^i_0 - \sum_j p_j z^i_j \geq 0$ ,  $i = 1, \dots, m$ , which are preferred by all agents above  $z^i$ .

The figure also suggests an iterative method to find a K-equilibrium. Let  $(L^A_1, L^B_2)$  with  $0 \leq L^A_1 \leq m^A_1$  and  $0 \leq L^B_2 \leq m^B_2$  be a rationing scheme on the

excess demands for A and B on the commodities 1 and 2 respectively. Then the new rationing scheme becomes  $(K^A_1, K^B_2)$  with  $K^A_1$ , the demand rationing for consumer A, equal to the assignment  $-m^B_1(L^B_2)$  of commodity 1 to consumer B and with  $K^B_2$  equal to  $-m^A_2(L^A_1)$ . Continuing this procedure leads to the K-equilibrium rationing scheme. However, generally, the iterative procedure  $(k^i, K^i) = (g^i(\tilde{e}), G^i(\tilde{e}))$  for all  $i$ , with  $\tilde{e} = (\tilde{e}^1, \dots, \tilde{e}^m)$  the effective excess demands obtained for the previous rationing schemes  $(l^i, L^i)$ ,  $i = 1, \dots, m$ , need not to converge to a set of K-equilibrium rationing schemes.

### 5. Typology of K-equilibria

In a K-equilibrium we have for each market that there is either rationing on the demands, or rationing on the supplies, or no rationing at all. So, for each market there are three possibilities. This implies that for an economy with  $n$  non-money commodities the number of different regimes equals  $3^n$  with  $2^n$  of them having rationing on all markets. When all prices of the non-money commodities are very high, we have in general excess supply on all markets. On the other hand, there will be excess demand on all markets if all prices are very low. Intermediate cases occur when some of the prices are relatively high, and the others are relatively low. For the case with  $n = 2$  the different regimes in the price space are sketched in Figure 5.1. In region I with  $p_1$  and  $p_2$  rather high, there is supply rationing on both markets, while in region III both markets are in excess demand. In region II (IV), commodity 2 is in excess demand (supply) and commodity 1 in excess supply (demand). On the boundary between two regions we have rationing on only one market. The prices  $p^W_1$  and  $p^W_2$  at the intersection point E of all regions are the Walrasian prices. For a numerical example we refer to Benassy (1975).

Insert Figure 5.1.

We will consider now the different regions in more detail. First we consider region I, in which there is supply rationing on both markets. If all agents are suppliers of both commodities, then no trade is feasible and a no trade equilibrium is achieved by setting  $l^i_1 = l^i_2 = 0$  for all  $i$ . A more interesting case occurs when some of the agents are suppliers of commodity 1 and demanders of commodity 2, say type A agents, and some other agents, say type B agents are demanders of commodity 1 and suppliers of commodity 2. Of course, for both commodities the total supply exceeds the total demand. Now, an equilibrium is achieved by setting rations  $l^i_1 (\leq 0)$  on the supplies of good 1 for the agents  $i$  of type A and rations  $l^i_2 (\leq 0)$  on the supplies of good 2 for the agents  $i$  of type B, such that the total effective demand of

the agents of type A (respectively B) for good 2 (respectively 1) equals the total rationed supply  $\sum_{i \in B} |l^i_2|$  of commodity 2 (respectively the total rationed supply  $\sum_{i \in A} |l^i_1|$  of commodity 1). Of course, the effective demand of agent  $i$  of type A (respectively B) for good 2 (respectively 1) depends on his ration  $l^i_1$  (respectively  $l^i_2$ ). Now, suppose agents of a third type, say type C, enter the scene with a demand for one of the commodities, say 1. The result is that agents of type A are able to sell more of commodity 1, so that their rations are weakened, i.e., the amounts  $|l^i_1|$  become greater. This increases the budget of the agents of type A for commodity 2, and hence the demands for commodity 2 will increase. Now, the agents of type B are able to sell more of commodity 2, and this will result in an additional demand for commodity 1 of the agents of type B. In this way we get the multiplier effect. The initial rationings on the supplies of commodity 1 have not only to be weakened to covering the demand of the agents of type C for commodity 1, but also to absorb the induced additional demand of the agents of type B for this commodity. Moreover, the initial rationings on the supplies of commodity 2 for the agents of type B have to be weakened to cover the induced additional demand for commodity 2 of the agents of type A. In cases of general excess supplies the government can therefore reduce the excess supplies (unemployment) by triggering off the multiplier effect through increasing their own demand for the commodities (i.e., extra government spendings). In the same way the government can reduce the excess demands by decreasing their demands if the economy is in a region III situation (general demand rationing).

We now consider the case that the economy is in a region II (or analogously region IV) situation. In region II there is supply rationing on commodity 1 and demand rationing on commodity 2. So, the agents of type A are rationed in their supplies of commodity 1 and in their demands for commodity 2, whereas the agents of type B are not rationed at all. In this case an additional demand of agents of type C for commodity 1 does not induce a multiplier effect. The only result is that the agents of type A are able to sell more of commodity 1. This results in an additional demand of these agents for commodity 2. However, the agents of type B are not constrained at all and therefore their supplies of commodity 2 do not change. So, the rationings on the demands of the agents of type A for commodity 2 have to stay on the same level. Therefore, the additional sale of commodity 1 by the agents of type A results in higher terminal holdings of money by these agents.

A specific example of a two non-money commodity economy is the Barro/Grossman-Malinvaud model. This model has been described in full detail in Malinvaud (1977). In this model the two commodities are consumption good (commodity 1) and labour (commodity 2). There is one producer, which is an agent of type A, supplying the consumption good and demanding for labour. The agents of



type B are the consumers, who supply labour and have a demand for the consumption good. Moreover the consumers have initial holdings of money. Money has utility as a means to transfer utility from the current period to the next period. The government with a demand for the consumption good is an agent of type C. In region I with roughly speaking a high (commodity) price  $p$  and a high (labour) wage  $w$ , the producer is constrained in his supply of the consumption good and the consumers are constrained in their supplies of labour (i.e., unemployed). This region is called the region of Keynesian Unemployment. The unemployment can be reduced by increasing the demand of the government for the consumption good. In region III, with a relatively low price and a low wage, the producer is constrained in his demand for labour and the consumers in their demand for the consumption good. In this region of Repressed Inflation the constraints can be weakened by decreasing the government demand of the consumption good. In region IV, with a low price for the consumption good and a high labour wage, the producer is unconstrained, whereas the consumers are constrained in the demand for the consumption good and in the supply of labour. In this case a change of the government demand does not affect the unconstrained producer and only has an impact on the rationing of the consumers. This region is called the region of Classical Unemployment. Because the producer can not be rationed simultaneously on his demand for labour and his supply of the consumption good we do not have region II in this model. On the boundary between the regions I and II the producer is constrained in either the demand for labour or the supply of the consumption good. In more complicated models, for instance a multi-period model in which the producer can keep consumption goods in stock, a non-empty region II is possible and is called the region of Under-Consumption. The typical partition of the price/wage space for the standard Malinvaud model without the possibility of underconsumption is sketched in Figure 5.2, in which  $p^*$  and  $w^*$  are the equilibrium prices.

Insert Figure 5.2.

## 6. Rationing schemes

Until now we did not specify the rationing schemes. It should be clear, however, that the equilibrium allocation depends on the specific rationing scheme. Drèze (1975), for instance, uses a uniform rationing scheme where the constraints on market  $j$  do not depend on the identity of the agent, i.e.,  $l^i = l$  and  $L^i = L$  for all  $i$ . Other rationing schemes may serve as well. For supply rationing, Kurz (1982) introduced the notion of fractional rationing, i.e., for all  $i$  and  $j$ ,  $l^i_j = -\alpha w^i_j$ , with 0



$\leq \alpha \leq 1$ . In this framework  $\alpha$  is the fraction of the initial endowment which can not be offered for sale. In a stochastic framework  $\alpha$  can be interpreted as the probability of becoming unemployed. For studies on stochastic rationing schemes we refer to e.g. Svensson (1980) and Wu (1985). Other examples of rationing are queueing and priority systems (see Benassy (1982, page 18)). Dräzen (1980) argues that it is of secondary importance which specific agents face rationing as long as we are concerned with the aggregate constraint. It should be noticed, however, that the allocation of the shortages is crucial in determining whether an equilibrium is socially acceptable or not.

Another example of rationing is proportional rationing (see Benassy (1982, page 19)). However, here the problem of manipulable rationing arises. A rationing scheme is non-manipulable if the assignments to agent  $i$  only depend on the expressed excess demands of the other agents. Under proportional rationing agents on the short side realize their demands, but agents on the long side receive an assignment proportional to their demand or supply. So, the assignment to agent  $i$  depends not only on the expressed excess demands of the other agents, but also on the excess demand expressed by himself. For example, for some market  $j$ , let  $d^i$  be the excess demand of agent  $i$ . Then  $D = \sum_i \max(0, d^i)$  is the total positive excess demand and  $S = \sum_i \max(0, -d^i)$  the total positive excess supply. Under proportional rationing consumer  $i$  receives in case  $D > S$

$$z^i = d^i \text{ if } d^i < 0 \text{ and } z^i = d^i S/D \text{ if } d^i \geq 0$$

and when  $D < S$

$$z^i = d^i \text{ if } d^i > 0 \text{ and } z^i = d^i D/S \text{ if } d^i \leq 0.$$

It is obvious that under proportional rationing an agent has the possibility to break the constraint by overbidding. If all agents behave in this way, no equilibrium may exist. Clearly, the proportional rationing is not consistent with Dräze's concept of constrained equilibrium. In the latter the agents maximize their utility under the ex-ante given rationing scheme, while under proportional rationing the scheme is determined ex-post depending on the actual demands and supplies. Proportional rationing, however, easily fits in Benassy's framework of a  $K$ -equilibrium, because in a  $K$ -equilibrium the transactions depend on the effective demands through the assignment functions  $F^i$ . Therefore, let us reconsider the  $K$ -equilibrium. In equilibrium, the actual transactions  $z^i_j = F^i_j(\tilde{e})$ ,  $j = 1, \dots, n$ , of agent  $i$  occur given the effective excess demands  $\tilde{e}^h$ ,  $h = 1, \dots, m$ . Moreover, the rationing schemes  $(l^i, L^i)$  satisfy  $l^i_j = z^i_j$  if  $z^i_j > \tilde{e}^i_j$  and  $L^i_j = z^i_j$  if  $z^i_j < \tilde{e}^i_j$ . Ex post, the realized

transactions maximize utility over the feasible transactions  $l_j^i \leq z_j^i \leq L_j^i$ ,  $j = 1, \dots, n$ . This characterization of a K-equilibrium shows that the binding constraints are determined through the assignment functions  $F_j^i$ . We have seen that given the rationing scheme  $(l^i, L^i)$  for all agents  $i$  the effective demand  $e^i(l^i, L^i)$  is an element of  $E^i(l^i, L^i)$ , i.e., the consumption realized through the expected transactions  $z_j^i(l^i, L^i) = \max(l_j^i, \min(\tilde{e}_j^i, L_j^i))$ ,  $j = 1, \dots, n$ , maximizes  $i$ 's utility. However, if the agent is aware of the fact that the rationing is determined through the assignments, he would not express an element of  $E^i(l^i, L^i)$  as his excess demand, but an excess demand  $\hat{e}^i$  such that the consumption induced by the assignments  $\hat{z}_j^i = F_j^i(\hat{e})$ ,  $j = 1, \dots, n$ , yields maximization of his utility, where  $\hat{e}^h$ ,  $h \neq i$ , are agent  $i$ 's expectations about the expressed demands of the other agents. In other words, agent  $i$  expresses effective demands  $\hat{e}_j^i$ ,  $j = 1, \dots, n$ , such that the consumption induced by

$$\hat{z}_j^i = \max(g_j^i(\hat{e}), \min(\hat{e}_j^i, G_j^i(\hat{e}))),$$

$j = 1, \dots, n$ , maximizes his utility. So, the agent will not maximize his utility under the given rationing scheme  $(l^i, L^i)$ , but under the expected new rationing scheme  $(g^i(\hat{e}), G^i(\hat{e}))$ . If the rationing functions  $g^i$  and  $G^i$  are manipulable, i.e., they depend on the effective demand  $\hat{e}^i$  expressed by agent  $i$ , and if all agents behave in this way, then the effective demands can not longer be restricted to belong to a bounded set and the fixed point argument can no be applied to prove Theorem 4.4. So, the rationing functions  $g^i$  and  $G^i$  should only depend on the expressed demands of the other agents.

## 7. Existence of constrained equilibria

In Section 4 the existence of a Drèze equilibrium, i.e., a constrained equilibrium without quantity constraints on the numeraire commodity, has been proved through proving the existence of a K-equilibrium. This proof uses a fixed point argument in the  $mn$ -dimensional space  $W^m$ . We noticed already that the iterative adjustment of the quantity constraints does not need to converge to an equilibrium. If not, we have to solve a system of equations to compute an equilibrium. In this case it is of great help to lower the dimension of the problem. Therefore we give a direct proof of the existence of a constrained equilibrium. The direct proof given in this section is based on the existence of fixed points in some appropriate  $(n+1)$ -dimensional set. Moreover, this type of proof allows for more general sets of admissible prices, while also the existence of other types of constrained equilibria can be proved, in particular the existence of a supply-

constrained equilibrium. Finally, we do not need to specify the constraint functions  $g^i$  and  $G^i$ .

Instead of commodity 0 we allow in this section for an arbitrarily chosen commodity as the numeraire commodity and hence we assume that the set of admissible prices is given by

$$P = \{p \in \Omega \mid 0 < p_j \leq \bar{p}_j < \infty \text{ for all } j\}.$$

For an a-priori chosen numeraire commodity  $h$  we say that a constrained equilibrium is a Drèze equilibrium with respect to  $h$  if commodity  $h$  is not rationed. Van der Laan (1980) considered a model without an a-priori chosen numeraire commodity. In this case the existence of a supply-constrained or unemployment equilibrium can be proved. If at an unemployment equilibrium commodity  $h$  is not rationed, then *ex post* commodity  $h$  can be chosen as the numeraire commodity and the equilibrium is *ex post* a supply-constrained Drèze equilibrium with respect to commodity  $h$ .

To prove the existence of a Drèze or an unemployment equilibrium we construct an excess demand function  $z$  for which each zero point yields a constrained equilibrium. Then we show that there do exist zero points of  $z$  for which the corresponding equilibria are Drèze or unemployment equilibria. For simplicity we assume uniform rationing schemes  $(I, L)$  in the remaining of this paper. Let  $Q$  be the  $(n+1)$ -dimensional subset of  $\Omega$  given by

$$Q = \{q \in \Omega \mid 0 \leq q \leq 2\bar{p}\}. \quad (7.1)$$

For  $q \in Q$ , let  $p(q) \in P$ ,  $l(q) \in -\Omega$ , and  $L(q) \in \Omega$  be defined by

$$p_j(q) = \max \{p_j, \min(\bar{p}_j, q_j)\}, \quad j = 0, \dots, n, \quad (7.2)$$

$$l_j(q) = -\min \{1, q_j/\bar{p}_j\}w_j, \quad j = 0, \dots, n, \quad (7.3)$$

$$L_j(q) = \min \{1, 2 - q_j/\bar{p}_j\}w_j, \quad j = 0, \dots, n. \quad (7.4)$$

The functions  $p_j(q)$ ,  $l_j(q)$ ,  $L_j(q)$  are illustrated in Figure 7.1. Observe that  $p_j \leq q_j \leq \bar{p}_j$  implies  $p_j(q) = q_j$ ,  $l_j(q) = -w_j$ , and  $L_j(q) = w_j$ . When  $q_j \geq \bar{p}_j$ , then  $p_j(q) = \bar{p}_j$ ,  $l_j(q) = -w_j$ , and  $L_j(q) = (2 - q_j/\bar{p}_j)w_j \leq w_j$ . When  $q_j \leq p_j$  then  $p_j(q) = p_j$ ,  $l_j(q) = -q_j w_j / p_j \geq -w_j$ , and  $L_j(q) = w_j$ .

Insert Figure 7.1.

Let  $x^i(q)$ ,  $i = 1, \dots, m$ , be the utility maximizing consumption of consumer  $i$  in the budget set

$$B^i(q) = \{x \in X^i \mid p^T(q)x \leq p^T(q)w^i, l(q) \leq x - w^i \leq L(q)\}$$

and let  $z(q) = \sum_i (x^i(q) - w^i)$ . From the assumptions  $A_1$ ,  $A_2$  and  $A_3$  it follows that  $x^i$  is a continuous function of  $q$  and satisfies  $p^T(q)x^i(q) = p^T(q)w^i$ ,  $i=1, \dots, m$ . Hence,  $z$  is a continuous function from  $Q$  into  $\mathbb{R}^{n+1}$  satisfying  $p^T(q)z(q) = 0$  for all  $q \in Q$ . Observe that  $q_j = 0$  implies  $l_j(q) = 0$  and hence  $z_j(q) \geq 0$ , while  $q_j = 2\bar{p}_j$  implies  $L_j(q) = 0$  and hence  $z_j(q) \leq 0$ .

**Lemma 7.1.** A zero point  $q^*$  of  $z$  induces a constrained equilibrium  $x^i(q^*)$ ,  $i=1, \dots, m$ ,  $p(q^*)$ ,  $l(q^*)$ , and  $L(q^*)$ .

**Proof:** By construction we have  $p(q^*) \in P$ ,  $l(q^*) \in -\Omega$ ,  $L(q^*) \in \Omega$ , and for all  $i$   $x^i(q^*) = d^i(p(q^*), l(q^*), L(q^*))$ . Further, since  $z(q^*) = 0$ , we obtain  $\sum_i x^i(q^*) = w$ . With  $z^i(q)$  the excess demand  $x^i(q) - w^i$  at  $q$  of agent  $i$ , we have that at a zero point  $q^*$  of  $z$  the excess demand of agent  $i$  of commodity  $j$  satisfies

$$-w_j < -w_j^i \leq z_j^i(q^*) \leq w_j - w_j^i < w_j.$$

By construction,  $l_j(q) > -w_j$  implies  $p_j(q) = p_j$  and  $L_j(q) = w_j$ , while  $L_j(q) < w_j$  implies  $p_j(q) = \bar{p}_j$  and  $l_j(q) = -w_j$ . Hence at a zero point  $q^*$  of  $z$ ,  $x^i(q^*)$ ,  $i = 1, \dots, m$ ,  $p(q^*)$ ,  $l(q^*)$ , and  $L(q^*)$  satisfy all the conditions of Definition 3.1.

If  $z(q^*) = 0$  and  $p_h \leq q_h^* \leq \bar{p}_h$ , then at the constrained equilibrium induced by  $q^*$  commodity  $h$  is not rationed. In fact  $q_h^* \leq \bar{p}_h$  implies that the demand of commodity  $h$  is not rationed, while  $q_h^* \geq p_h$  implies that the supply is not rationed. Hence we have the following corollaries.

**Corollary 7.2.** When for some  $h$ ,  $h \in \{0, \dots, n\}$ ,  $p_h \leq q_h^* \leq \bar{p}_h$ , then a zero point  $q^*$  of  $z$  yields a Dr ze equilibrium with respect to commodity  $h$ .

**Corollary 7.3.** A constrained equilibrium induced by a zero point  $q^*$  of  $z$  yields an unemployment equilibrium if there exists an  $h$  with  $p_h \leq q_h^* \leq \bar{p}_h$  and  $q_j^* \leq \bar{p}_j$  for all  $j \neq h$ .



The next lemma says that any one of the counting variables  $q_j$  can be chosen freely, in the sense that if one of these variables is given an a-priori value, there exists a zero point of  $z$  for which that variable has this value. In fact, if for some  $j$ ,  $q_j$  is fixed, then it is possible to determine the other variables  $q_h$ ,  $h \neq j$ , such that all markets  $h \neq j$  are in equilibrium. Since for all  $q$  we have  $p^T(q)z(q) = 0$  (Walras' law), market  $j$  must be in equilibrium if all markets  $h \neq j$  are in equilibrium.

**Lemma 7.4.** For each  $\hat{q}_h$ ,  $0 \leq \hat{q}_h \leq 2\bar{p}_h$ ,  $h \in \{0, \dots, n\}$ , there exists a zero point  $q^*$  of  $z$  such that  $q_h^* = \hat{q}_h$ .

**Proof:** Let the continuous function  $f$  from  $Q$  to  $Q$  be defined by

$$f_j(q) = \max(0, \min(2\bar{p}_j, q_j + z_j(q))), \quad j \neq h,$$

and  $f_h(q) = \hat{q}_h$ . According to Brouwer's fixed point theorem there exists a point  $q^*$  in  $Q$  such that  $f(q^*) = q^*$ . Clearly,  $q_h^* = \hat{q}_h$ . Suppose that for some  $j \neq h$ ,  $q_j^* = 2\bar{p}_j$  (respectively  $= 0$ ), then  $z_j(q^*) \leq 0$  (respectively  $\geq 0$ ), and hence  $f_j(q^*) = q_j^*$  implies  $z_j(q^*) = 0$ . When for some  $j \neq h$ ,  $0 < q_j^* < 2\bar{p}_j$ , then  $f_j(q^*) = q_j^* + z_j(q^*)$  and hence  $z_j(q^*) = 0$ . Consequently, the fixed point  $q^*$  of  $f$  implies that  $z_j(q^*) = 0$  for all  $j \neq h$ . Together with Walras' law and the fact that  $p_h > 0$  this implies that also  $z_h(q^*) = 0$  and hence  $q^*$  is a zero point of  $z$ .

**Theorem 7.5.** For each  $h$ , there exists a Drèze equilibrium with respect to commodity  $h$ .

**Proof:** Set  $\hat{q}_h$  such that  $p_h \leq \hat{q}_h \leq \bar{p}_h$ . The theorem follows from Corollary 7.2 and Lemma 7.4.

The existence of an unemployment equilibrium follows immediately from the next lemma, which says that for each a-priori chosen  $\delta$ ,  $0 \leq \delta \leq 1$ , there exists a zero point  $q^*$  of  $z$  on the upper boundary of the box

$$Q_\delta = \{q \in Q \mid \max_h q_h/2\bar{p}_h \leq \delta\}.$$

Thus,  $\max_h q_h/2\bar{p}_h$  is now taken fixed instead of giving one of the variables an a priori chosen value.

**Lemma 7.6.** For each  $\delta$ ,  $0 \leq \delta \leq 1$ , there exists a zero point  $q^*$  in  $Q$  of  $z$  such that  $\max_j q_j^*/2\bar{p}_j = \delta$ .

Proof: For  $\delta = 1$ ,  $q^* = 2\bar{p}$  yields the trivial equilibrium  $x^i(q^*) = w^i$  for all  $i$ ,  $l(q^*) = -w$ ,  $L(q^*) = 0$ , and  $p(q^*) = \bar{p}$ , and hence  $z(q^*) = 0$ . So, suppose that  $0 \leq \delta < 1$ . For all  $q \in Q_\delta$ , let  $r(q)$  be the intersection of the set  $\{q \in Q \mid \max_j q_j/2\bar{p}_j = \delta\}$  and the line segment from  $q$  to  $2\bar{p}$ . So,  $r(q) = (1-\lambda(q))q + \lambda(q)2\bar{p}$ , where

$$\lambda(q) = (\delta 2\bar{p}_h - q_h)/(2\bar{p}_h - q_h)$$

with  $h$  an index such that  $q_h/2\bar{p}_h = \max_j q_j/2\bar{p}_j$ , see Figure 7.2.

Insert Figure 7.2.

By construction  $r(q) = q$  if  $q_h/2\bar{p}_h = \delta$  and  $r_j(q) > q_j$  for all  $j$  if  $q_h/2\bar{p}_h < \delta$ .

Let the continuous function  $f$  from  $Q_\delta$  to  $Q_\delta$  be defined by

$$f_j(q) = \max(0, \min(\delta 2\bar{p}_j, r_j(q) + z_j(q))), \quad j = 0, \dots, n.$$

Again according to Brouwer's fixed point theorem there exists a point  $q^*$  in  $Q_\delta$  such that  $f(q^*) = q^*$ . Suppose that  $\max_j q_j^*/2\bar{p}_j < \delta$ , and hence  $r_j(q^*) > q_j^*$  for all  $j$ . Because  $q_j^* = f_j(q^*)$  this implies  $z_j(q^*) < 0$  for all  $j$ . Since  $p_j(q^*) \geq \bar{p}_j > 0$  for all  $j$  this contradicts Walras' law. Hence the fixed point  $q^*$  of  $f$  satisfies  $\max_j q_j^*/2\bar{p}_j = \delta$ . It remains to prove that  $z_j(q^*) = 0$  for all  $j$ . Since  $\max_j q_j^*/2\bar{p}_j = \delta$  implies  $r(q^*) = q^*$ , we obtain

$$q_j^* = \max(0, \min(\delta 2\bar{p}_j, q_j^* + z_j(q^*))), \quad j = 0, \dots, n.$$

If for some  $j$  we have  $q_j^* = 0$ , then  $z_j(q^*) \geq 0$  and hence  $f_j(q^*) = q_j^*$  implies  $z_j(q^*) = 0$ . If  $0 < q_j^* < \delta 2\bar{p}_j$ , then  $f_j(q^*) = q_j^*$  implies  $z_j(q^*) = 0$ . Finally, if  $q_j^* = \delta 2\bar{p}_j$ , then  $f_j(q^*) = q_j^*$  implies  $z_j(q^*) \geq 0$ . Again from Walras' law and the fact that  $p_j(q^*) > 0$  this implies that also  $z_j(q^*) = 0$  if  $q_j^* = \delta 2\bar{p}_j$ , and hence  $q^*$  is a zero point of  $z$ .

Theorem 7.7. There exists an unemployment equilibrium, such that there is an unrationed commodity  $h$  with price equal to  $\bar{p}_h$ .

Proof: Take  $\delta = 1/2$ . From Lemma 7.6 it follows that there exists a zero point  $q^*$  of  $z$  such that  $\max_j q_j^*/2\bar{p}_j = 1/2$ . So, at  $q^*$  there exists an index  $h$  such that  $q_h^*/2\bar{p}_h = 1/2$ , implying that  $q_h^* = \bar{p}_h$  and hence  $p_h(q^*) = \bar{p}_h$ . Moreover, we have  $q_j^* \leq \bar{p}_j$  for all  $j \neq h$ , implying that there is no rationing on the demands.

It should be observed that the rationing is completely determined by  $q$  through (7.3) and (7.4). By taking continuous individual constraint functions  $l_j^i(q)$  and  $L_j^i(q)$  satisfying  $l_j^i(q) = 0$  if  $q_j = 0$ ,  $l_j^i(q) \leq -w_j^i$  if  $q_j \geq \bar{p}_j$ ,  $L_j^i(q) = 0$  if  $q_j = 2\bar{p}_j$ , and  $L_j^i(q) \geq w_j^i$  if  $q_j \leq \bar{p}_j$ , the existence of constrained equilibria can be proved under non-uniform rationing.

As mentioned already in the proof of Lemma 7.6, we have that for  $\delta = 1$ ,  $q^* = 2\bar{p}$  yields the trivial equilibrium with  $L = 0$ , i.e., with complete rationing of all the demands. Also, for  $\delta = 0$ ,  $q^* = 0$  is the unique element in  $Q_0$  and yields the trivial equilibrium with complete rationing of all the supplies.

We have seen that due to Walras' law there is one degree of freedom in the set  $Q$  of variables determining the prices and the rationing schemes. When defining the function  $h$  from  $Q$  to  $\mathbb{R}^{n+1}$  by  $h_j(q) = p_j(q)z_j(q)$ ,  $j = 0, \dots, n$ , we have that  $h$  and  $z$  are equivalent in the sense that  $z(q^*) = 0$  if and only if  $h(q^*) = 0$ . Clearly, because of Walras' law,  $\sum_j h_j(q) = 0$  and hence  $h$  is a continuous function from the  $(n+1)$ -dimensional set  $Q$  to the  $n$ -dimensional set

$$S = \{y \in \mathbb{R}^{n+1} \mid \sum_j y_j = 0\}.$$

In general we have because of the implicit function theorem that for a continuously differentiable function  $h: Q \rightarrow S$  and a regular value  $c \in S$  of  $h$ , the set  $h^{-1}(c)$  is a disjoint union of paths and loops. Each path or loop is a one-dimensional manifold. A loop is a closed cycle and has no end points, while a path has two end points on the boundary of  $Q$ . We have seen that the function  $h: Q \rightarrow S$  has two zero points on the boundary of  $Q$ , namely  $q^* = 0$  and  $q^* = 2\bar{p}$ , corresponding to the two trivial equilibria. Now, assume that  $z_j(q) > 0$  if both  $q_j = 0$  and there exists an  $h$  with  $q_h > 0$ . This assumption says that there is a positive excess demand for commodity  $j$  if it cannot be offered for sale ( $q_j = 0$  implies  $l_j^i(q) = 0$ ) and at least one other commodity can be offered for sale. So, at least one consumer wants to buy some amount of commodity  $j$ . Also, assume that  $z_j(q) < 0$  if both  $q_j = 2\bar{p}_j$  and there exists an  $h$  with  $q_h < 2\bar{p}_h$ . Under these assumptions  $q^* = 0$  and  $q^* = 2\bar{p}$  are the only two zero points of  $h$  on the boundary of  $Q$ . So, if  $0 \in S$  is a regular value of  $h$  and  $h$  is continuously differentiable, then  $h^{-1}(0)$  contains just one path. This path runs from  $0$  to  $2\bar{p}$ . The existence of such a path proves immediately both Lemma 7.4 and Lemma 7.6. Indeed, a path from  $q = 0$  to  $q = 2\bar{p}$  crosses at least once the set  $\{q \in Q \mid q_h = \hat{q}_h\}$  for any  $0 \leq \hat{q}_h \leq 2\bar{p}_h$ ,  $h \in \{0, \dots, n\}$ , as well as the upper boundary of  $Q_\delta$  for any  $0 \leq \delta \leq 1$ . However, the reverse is not true, i.e., the Lemmas 7.4 and 7.6 do not prove the existence of a path connecting  $0$  and  $2\bar{p}$ . On the contrary, there are severe difficulties in proving the existence of such a path. From Sard's

theorem it follows that if  $h$  is continuously differentiable almost every  $c \in S$  is a regular value of  $h$  and hence we could assume that the vector of zeros is a regular value of  $h$ . However,  $h$  is in general not continuously differentiable. This follows immediately from the fact that  $p_j(q)$  is not differentiable at points  $q$  where  $q_j$  equals  $\underline{p}_j$  or  $\bar{p}_j$ . Moreover,  $z_j(q)$  is not differentiable at a point  $q$  at which a consumer switches from not being rationed to becoming rationed in commodity  $j$ . Nevertheless, using simplicial approximation theory (e.g. see van der Laan (1982) and van der Laan and Talman (1987)) we can prove the following theorem.

**Theorem 7.8.** For any  $\epsilon > 0$ , there exists a path of points  $H_\epsilon$  in  $Q$  with end points  $q = 0$  and  $q = 2\bar{p}$ , such that

$$\max_j |h_j(q)| < \epsilon$$

for any  $q \in H_\epsilon$ .

So, for any  $\epsilon > 0$ , there exists a path of approximating zero points of  $h$  with accuracy equal to  $\epsilon$ . Since  $|h_j(q)| < \epsilon$  implies  $|z_j(q)| < \epsilon/\underline{p}_j$  it follows that there is a path of approximating zero points of  $z$  connecting 0 and  $2\bar{p}$ , and hence a path of approximating constrained equilibria connecting the two trivial equilibria. In the sequel, if we speak about the path  $H$  of (constrained) equilibria we mean the path of approximating equilibria induced by the points on the path  $H_\epsilon$  of approximating zero points of  $z$  for an arbitrarily small  $\epsilon > 0$ .

### 8. Supply-constrained versus demand-constrained equilibria

In the previous section we proved the existence of an unemployment or supply-constrained equilibrium. Analogously the existence of a demand-constrained equilibrium can be proved. Instead of doing so, let us consider the path  $H$  of equilibria connecting the two trivial equilibria. Without loss of generality, we say that there is supply rationing on commodity  $j$  if  $q_j < \underline{p}_j$  and hence  $I_j > -w_j$  and that there is demand rationing on commodity  $j$  if  $q_j > \bar{p}_j$  and hence  $L_j < w_j$ . Observe that commodity rationing defined in this way does not imply rationing of the consumers in that commodity; it only means that it may occur that a consumer is rationed in that commodity.

Going along the path  $H$  from 0 to  $2\bar{p}$  we have that for each  $j$  the variable  $q_j$  goes from 0 to  $2\bar{p}_j$ , all passing the interval  $[\underline{p}_j, \bar{p}_j]$ . If  $q_j < \underline{p}_j$  we have that  $I_j > -w_j$  and  $L_j = w_j$ , and hence commodity  $j$  is rationed on the supply side. Furthermore,  $\underline{p}_j \leq q_j \leq \bar{p}_j$  implies that  $I_j = -w_j$  and  $L_j = w_j$ , and hence commodity



$j$  is not rationed at all. Finally, for  $q_j > \bar{p}_j$  we have that  $l_j = -w_j$  and  $L_j < w_j$ , so that commodity  $j$  is rationed on the demand side. So, for all  $q$  on the path close to 0, namely for  $q$  with  $q_j < \underline{p}_j$  for all  $j$ , we have supply rationing on each market, while for all  $q$  close to  $2\bar{p}$ , namely with  $q_j > \bar{p}_j$  for all  $j$ , we have demand rationing on each market. Going along the path from 0 to  $2\bar{p}$  each market switches therefore first from supply rationing to no rationing and eventually to demand rationing. Going along the path  $H$  and starting at  $q = 0$ , let  $h$  be the first index for which  $q_h$  becomes equal to  $\underline{p}_h$ , say at point  $q^-$ . Then  $q^-$  yields a supply-constrained equilibrium with no rationing of commodity  $h$ . This equilibrium is also a Drèze equilibrium with commodity  $h$  as the ex post chosen numeraire. On the other hand, let  $q^+$  be a point on the path such that there is an index  $k$  with  $q_k = \bar{p}_k$  and  $q_j \geq \bar{p}_j$  for all  $j \neq k$ . It follows from the discussion above that such a point exists on the path of equilibria. Then  $q^+$  yields a demand-constrained equilibrium with commodity  $k$  unrationed <sup>4)</sup>. It is also a Drèze equilibrium with commodity  $k$  as the ex post chosen numeraire. As argued above, for any  $j$  there exists a point  $q$  on the path such that  $\underline{p}_j \leq q_j \leq \bar{p}_j$ . Such a point yields a Drèze equilibrium with respect to commodity  $j$  as the unrationed ex post chosen numeraire commodity. For all other commodities there is either demand rationing or supply rationing or neither. For  $j = h$  we have an equilibrium with supply rationing on the other commodities, for  $j = k$  we have an equilibrium with demand rationing on the other commodities <sup>5)</sup>. Roughly speaking, we may say that, relative to the Walrasian price system, commodity  $h$  has the lowest price and commodity  $k$  has the highest price. To conclude this discussion, let us consider the case that along the path  $H$ ,  $q_j$  increases monotonically for all  $j$ , and that at any point  $q$  along  $H$  at most one variable  $q_j$  lies in the interval  $[\underline{p}_j, \bar{p}_j]$ . The latter is in general true if for all  $j$ ,  $\underline{p}_j$  is close or equal to  $\bar{p}_j$ . In this case there is an ordering  $i_0 (=h), i_1, \dots, i_{n-1}, i_n (=k)$  of the indices  $0, \dots, n$ , such that for all  $j$  at the induced Drèze equilibrium with respect to commodity  $i_j$ , the commodities  $i_{j+1}, \dots, i_n$  are supply constrained and the commodities  $i_0, \dots, i_{j-1}$  are demand constrained, so that along the path  $H$  all markets switch successively in this order from supply to demand rationing.

The motivation behind the proof of a supply-constrained equilibrium lies in the idea that demand rationing rarely occurs in market economies whereas supply rationing is very common. Clearly, pure existence does not explain why supply-constrained equilibria should occur more frequently than demand-constrained equilibria. We return to this topic in Section 10. Another problem in the theory of

<sup>4)</sup> A formal proof of the existence of a demand-constrained equilibrium goes along the lines of the existence proof of a supply-constrained equilibrium.

<sup>5)</sup> However, observe that nothing has been said about unicity, so that also other supply-constrained or demand-constrained equilibria may occur.

supply-constrained equilibria has been raised by Dehez and Drèze (1984). They argue that in an economy with money, it is more realistic to choose money a priori as the unrationed numeraire commodity, since quantity constraints on net trades of money are very rarely observed. By definition, a supply-constrained equilibrium excludes the possibility of an a priori chosen numeraire commodity. In other words, rationing of money may happen in a supply-constrained equilibrium. Dehez and Drèze however provided sufficient conditions for the existence of a supply-constrained equilibrium with no rationing on an a priori chosen always desired numeraire (money). In fact, it is sufficient and necessary to allow for flexible money prices.

## 9. Flexible money prices

In this section we consider a simplified version of the Dehez-Drèze model <sup>6)</sup> (see also van der Laan (1984)). This simplification does not affect the generality of the results. In the model, the relative prices of the non-money commodities are bounded by upper and lower bounds. In the extreme case that the upper bounds are equal to the lower bounds, we have that for each pair of non-money commodities the ratio of the prices is fixed, but the level of these prices with respect to the price of money is not fixed. The price level of the non-money commodities is determined by a price index, which is determined by the prices of the index commodities. This set of index makers is a subset of the set of commodities other than money. Under certain restrictions on the prices of the index commodities, there exists a supply-constrained equilibrium such that money is not rationed.

The money commodity, indexed by  $j = 0$ , is again used as the numeraire commodity and its price is set equal to 1. The set  $I$  of index commodities is a subset of  $\{1, \dots, n\}$  and defines a price index  $\pi(p) = \pi(p_j, j \in I)$ . This index determines the level of the prices of all the non-money commodities  $j$ ,  $j = 1, \dots, n$ . The price index function  $\pi$  is assumed to be continuous in  $p$  and homogeneous of degree one, i.e.,  $\pi(\alpha p) = \alpha \pi(p)$  for all  $\alpha > 0$ . The relative prices of the non-money commodities, i.e., the ratios between the prices of the commodities and the price index, are restricted. So, the set  $P^*$  of admissible prices is given by

$$P^* = \{p \in \Omega \mid p_0 = 1, \pi(p) \underline{p}_j \leq p_j \leq \pi(p) \bar{p}_j, j=1, \dots, n\},$$

with  $0 < \pi(\underline{p}_j, j \in I) \leq 1 \leq \pi(\bar{p}_j, j \in I)$  and  $0 < \underline{p}_j \leq \bar{p}_j < \infty$  for all  $j$ . The latter restrictions ensure that  $P^*$  is not empty.

<sup>6)</sup> In the model of Dehez and Drèze there is also a production sector.

As an example, let  $n = 2$  with commodity  $j = 1$  labour and commodity  $j = 2$  a consumption good. In case of price indexation for the wages the level of the wages will depend on the price of the consumption good. We have then that  $I = \{2\}$ , and for instance the price index  $\pi(p) = p_2$ . Hence the price set  $P^*$  is equal to

$$P^* = \{p \in R_+^3 \mid p_0 = 1, p_2 \bar{p}_1 \leq p_1 \leq p_2 \bar{p}_1\},$$

i.e., the ratio between the price of labour and the price of the consumption good is bounded, and fixed if  $\bar{p}_1 = \bar{p}_1$ . Observe that the restriction  $p_2 \bar{p}_1 \leq p_1 \leq p_2 \bar{p}_1$  is redundant, because the non-emptiness of  $P^*$  requires that  $\bar{p}_2 \geq 1$  and  $\bar{p}_2 \geq 1$ .

Normalizing the prices by setting  $\pi(p) = 1$  instead of  $p_0 = 1$  we obtain the set of prices

$$P' = \{p \in \Omega \mid p_0 \geq 0, p_j \leq \bar{p}_j, j=1, \dots, n, \text{ and } \pi(p) = 1\}.$$

Clearly, because of the homogeneity of  $\pi$  we have that if  $\pi(p) > 0$  then  $p \in P^*$  implies that  $\hat{p} = \pi^{-1}(p) \in P'$ . Also, if  $p_0 > 0$  then  $p \in P'$  implies  $p^* = p_0^{-1}p \in P^*$ , since  $\pi(p_0^{-1}p) = p_0^{-1}$  by the fact that  $\pi$  is homogeneous and  $\pi(p) = 1$  for  $p \in P'$ .

We have seen in the previous sections that in a constrained equilibrium supply rationing is only allowed if the price is on its lower bound. However, without further assumptions this complementarity condition between rationing and price binding can not be guaranteed to hold when dealing with the set  $P'$ . Since  $\pi(p)$  is restricted to be equal to one, the prices of the index makers can not reach the lower bounds simultaneously if  $\pi(\bar{p}_j, j \in I) < 1$ . So, excess supply can not enforce minimum prices for all index makers simultaneously, unless  $\pi(\bar{p}_j, j \in I) = 1$ . Therefore in the following we consider the case that the lower bounds  $\bar{p}_j$  satisfy  $\pi(\bar{p}_j, j \in I) = 1$ . Assuming that  $\pi(p) > \pi(\bar{p})$  if  $p \geq \bar{p}$  and  $p_j > \bar{p}_j$  for at least one  $j \in I$ ,  $\bar{p} \geq \bar{p}$  and  $\pi(p) = 1$  for all  $p$  implies that  $\bar{p}_j = \bar{p}_j$  for all  $j \in I$ , and hence that the index makers have fixed prices  $p_j = \bar{p}_j = \bar{p}_j$ .

Under the assumptions of Section 2, we are now able to state the following result, which strengthens the result of Dehez and Drèze in the sense that they proved Theorem 9.1 with in iii)  $l_j < 0$  instead of  $l_j = -w_j$ . Recall that in an unemployment equilibrium there is no demand rationing, i.e.,  $L_j = \infty$  for all  $j$ .

**Theorem 9.1.** If  $\bar{p}_j = \bar{p}_j$  for all  $j \in I$ , then there is an unemployment equilibrium with allocation  $x^i$ ,  $i = 1, \dots, m$ , a rationing scheme  $l \leq 0$ , and a price  $p \in P'$ , such that

- i)  $l_0 = -w_0$ ,
- ii) for all  $j$ ,  $p_j > \bar{p}_j$  implies  $l_j = -w_j$ ,



iii) for at least one  $j \neq 0$ ,  $p_j = \bar{p}_j$  and  $l_j = -w_j$ .

For the proof of this theorem we refer to van der Laan (1984).

The theorem says that in case of fixed prices of the index makers there exists a supply-constrained equilibrium with no rationing on the money commodity and no rationing on at least one other commodity. In fact, there is an equilibrium in which the price of an unrationed non-money commodity is on its upper bound.

The opposite case of fixed prices for the index commodities is the case that the prices of the index commodities are free, i.e.,  $p_j = 0$  and  $\bar{p}_j = \infty$  for all  $j \in I$ . However, to prove existence in this case, we need some restrictions on the price index function  $\pi(p)$  (see Weddepohl (forthcoming) for a detailed discussion). Here we restrict ourselves to the case that  $\pi(p) = \sum_{j \in I} p_j$ . The set of admissible prices then becomes

$$P'' = \{p \in \Omega \mid p_0 \geq 0, \sum_{j \in I} p_j = 1, p_j \leq \bar{p}_j, j \notin I \cup \{0\}\}.$$

with  $p_j > 0$  for all  $j \notin I \cup \{0\}$ . For technical reasons we assumed until now that  $p_j > 0$  for all  $j$ . This assumption guarantees that  $-p^T l$  is positive for all  $l \leq 0$  with  $l_j < 0$  for at least one non-money commodity  $j$ . By assumption  $A_3$  of Section 2 this implies that for all  $i$ ,  $\sum_j p_j \min(w_j^i, -l_j) > 0$ , i.e., each consumer has a positive income, which is a necessary condition for the continuity of the excess demand functions. However, to guarantee the continuity of the excess demand functions in the present case it is sufficient that  $-p^T l > 0$  for all  $l \leq 0$  with  $l_j < 0$  for some  $j \notin I \cup \{0\}$  or  $l_j < 0$  for all  $j \in I$ , which will be true if  $p_j > 0$  for all  $j \notin I \cup \{0\}$  and  $\sum_{j \in I} p_j = \pi(p) = 1$  for all  $p$ . We now have the following theorem.

**Theorem 9.2.** There is an unemployment equilibrium with allocation  $x^i$ ,  $i = 1, \dots, m$ , a rationing scheme  $l \in -\Omega$ , and a price  $p \in P''$ , such that

- i)  $l_0 = -w_0$ ,
- ii) for all  $j \notin I \cup \{0\}$ ,  $p_j > 0$  implies  $l_j = -w_j$ ,
- iii) there is an index  $j \notin I \cup \{0\}$  with  $p_j = \bar{p}_j$  and  $l_j = -w_j$ , or for all  $j \in I$ ,  $l_j = -w_j$ .

For the proof we refer again to van der Laan (1984).

The theorem says that in the case of free prices for the index makers there exists a supply-constrained equilibrium such that money is not rationed and moreover at least one of the non-index commodities is not rationed or all index makers are not

rationed. From the viewpoint of rationing, the index makers can be seen as a composite commodity with price  $\pi(p) = 1$ . As long as all the non-index commodities and some of the index makers are rationed, adjusting of the prices and amounts of rationing is possible until all index makers are not rationed anymore or at least one of the non-index makers is unrationed and has its price on the upper bound.

Kurz (1982) considered a set of admissible prices  $P^m$  defined as

$$P^m = \{p \in \Omega \mid p_0 + \sum_{j \in I} p_j = 1, p_j = \phi_j(p_h, h \in I), j \notin I \cup \{0\}\}.$$

If for  $j \notin I \cup \{0\}$ ,  $p_j = \bar{p}_j = \hat{p}_j$ , and when  $\phi_j(p_h, h \in I) = \hat{p}_j \sum_{h \in I} p_h$ , the sets  $P''$  and  $P^m$  are equivalent since  $p/(p_0 + \sum_{h \in I} p_h)$  is in  $P^m$  if  $p \in P''$  and  $p/(\sum_{h \in I} p_h)$  is in  $P''$  if  $p \in P^m$ . Kurz proved the existence of an unemployment equilibrium with possibly rationing on the supply of money. The formulation of  $P''$  instead of  $P^m$  however, enables us to exclude rationing of money supplies. So, Theorem 9.2 strengthens Kurz's result by stating that money is not rationed. On the other hand, it extends the result of Dehez and Drèze to economies with free relative prices of the index makers.

Further results on supply-constrained equilibria can be found in Wu (1985) and Weddepohl (forthcoming).

#### 10. A rationale for supply-constrained equilibria

In Section 8 we showed that in a non-money economy each commodity can be chosen as the numeraire commodity for a Drèze equilibrium. If along the path  $H$  of equilibria in the set  $Q$  for all  $j$  the variable  $q_j$  increases monotonically, there is an ordering  $i_0, i_1, \dots, i_{n-1}, i_n$  of the indices  $0, \dots, n$ , such that for all  $j$  at the Drèze equilibrium with respect to commodity with index  $i_j$  as the numeraire commodity, the commodities  $i_{j+1}, \dots, i_n$  are not demand constrained and the commodities  $i_0, \dots, i_{j-1}$  are not supply constrained. This does not explain why supply-constrained equilibria should occur more frequently than demand-constrained equilibria. However, the situation differs for a money economy with flexible money prices. To illustrate this we restrict ourselves to the case that for all  $j$ ,  $p_j = \bar{p}_j = p^*_j$ . Since  $\underline{p}$  and  $\bar{p}$  satisfy  $\pi(\underline{p}) \leq 1$  and  $\pi(\bar{p}) \geq 1$  we have that  $\pi(p^*) = 1$  and hence the price set  $P''$  defined in Section 9 becomes

$$P' = \{p \in \Omega \mid p_0 \geq 0, p_j = p^*_j, j=1, \dots, n\}.$$

Normalizing  $p_0$  to be equal to zero we obtain

$$P^* = \{p \in \Omega \mid p_0 = 1 \text{ and for some } \alpha \geq 0, p_j = \alpha p_j^*, j = 1, \dots, n\},$$

Theorem 9.1 says that for this set of prices there exists a supply-constrained equilibrium with no rationing on the money commodity and no rationing on at least one non-money commodity. On the other hand, it is easy to see that for  $\alpha$  close to zero all markets will be in excess demand. This is illustrated in Figure 10.1, in which the set of prices  $P^*$  has been drawn in the partitioned space of prices  $(p_1, p_2)$  according to Figure 5.1. Figure 10.1 shows that the economy is in the regime of

- a) demand rationing on both markets for  $\alpha < \alpha^1$ ;
- b) demand rationing on market 1 and no rationing on market 2 for  $\alpha = \alpha^1$ ;
- c) demand rationing on market 1 and supply rationing on market 2 for  $\alpha^1 < \alpha < \alpha^2$ ;
- d) no rationing on market 1 and supply rationing on market 2 for  $\alpha = \alpha^2$ ;
- e) supply rationing on both markets for  $\alpha > \alpha^2$ .

Insert Figure 10.1.

The existence of an equilibrium according to case d) has been proved in Theorem 9.1<sup>7)</sup>. However, we see that for case b) we have the analogous result for a demand-constrained equilibrium. In fact, going from very low values of  $\alpha$  to very high values of  $\alpha$  all markets switch from demand constrained markets to supply constrained markets. In case that the number of markets with supply rationing is non-decreasing when  $\alpha$  increases, there is an ordering  $i_1, i_2, \dots, i_n$  of the indices  $1, \dots, n$  and an increasing sequence of positive numbers  $\alpha_k$ ,  $k = 1, 2, \dots, n$ , such that at prices  $p_j = \alpha_k p_j^*$ ,  $j = 1, \dots, n$ , market  $i_k$  is unrationed, the markets  $i_1, \dots, i_{k-1}$  are supply constrained, and the markets  $i_{k+1}, \dots, i_n$  are demand constrained. So, again demand rationing is as reasonable as supply rationing. However, when the ratio between the prices of the non-money commodities are fixed, but the price level is flexible, it is not unlikely that the price level will move upwards as long as some markets are in excess demand. This corresponds to the idea that downward price rigidities are stronger than upward price rigidities. Indeed, the economy tends to a supply-constrained equilibrium if the price level goes upwards under excess demands for some commodities. This explanation requires to reconsider the partitioning of the non-money commodities in price making and price following commodities. When

<sup>7)</sup> In case of the Malinvaud model with a fixed ratio between the price of the consumption good and the wage of labour, such a supply-constrained equilibrium lies either on the border between the regions of classical and Keynesian unemployment with the consumers constrained in their supply of labour, or on the boundary between the regions of repressed inflation and Keynesian unemployment with the producer constrained in his supply of the consumption good.

assuming that the price level goes up as long as there is demand rationing, the commodities in excess demand are the price makers and the commodities in excess supply are the price following commodities.

### 11. Unemployment compensation

We have seen that in an economy with non-Walrasian prices equilibrium of demand and supply can be reached through supply rationing. Indeed, in the real world supply rationing frequently occurs. Unemployment is a well-known example. However, knowing that there exists an unemployment equilibrium does not help unemployed people very much. To be unemployed has very serious social and economic impacts. The impact of unemployment on social life is hardly to overcome. In fact, in many cases only getting a job helps. In this sense job sharing may be of great help. The distribution of jobs over people is determined by the rationing scheme. In our framework we call someone unemployed (or supply-constrained) on market  $h$  if he wants to work more (or if he wants to sell more) than the rationing amount  $|l_h^s|$ . So, we do not make a difference between the case of total unemployment (i.e.,  $l_h = 0$ ) and the case of underemployment (i.e.,  $l_h \neq 0$ ). In both cases there is a loss of utility due to the constraint. Of course, the loss of utility in the case of unemployment will be higher than the loss of utility in the case of underemployment. In this section we consider a model in which this loss of utility is compensated by unemployment doles or other subsidies for unemployed people. This results in a model of an economy in which an agent perceives quantity constraints on his net sales, and receives an unemployment compensation for the loss of utility due to the constraints on the supplies. The unemployment compensations are financed by levying taxes. We assume that there is some institution or a government which collects the taxes and distributes the tax revenues among unemployed consumers through lump-sum compensations. It is not our purpose to discuss tax models, but we are only concerned about the existence of equilibria with unemployment compensations. We first consider a model in which the tax is only levied on net purchases.

Let us consider an economy with  $n+1$  commodities and a set of admissible prices

$$P = \{p \in \Omega \mid p_j \leq \bar{p}_j \text{ for all } j\},$$

with for all  $j$ ,  $0 < p_j \leq \bar{p}_j < \infty$ , i.e., an economy without money or an a priori chosen numeraire commodity. To finance unemployment compensations we introduce taxes on net purchases. So, a tax vector is a vector  $t \in \Omega$ . Given a consumption  $x^i$ ,



the tax to be paid by consumer  $i$  equals  $t^T(x^i - w^i)^+ = \sum_j t_j (x^i - w^i)^+_j$  with  $a^+_j = \max(0, a_j)$ . Given a price  $p \in P$ , a rationing scheme  $l \leq 0$ , an excise tax vector  $t \geq 0$ , and a lump-sum compensation  $s^i$ , the budget set of consumer  $i$  becomes

$$B^i(p, l, t, s^i) = \{x \in X^i \mid p^T x + t^T(x - w^i)^+ \leq p^T w^i + s^i, l \leq x - w^i\}.$$

So, the consumer  $i$  is rationed in his supplies, pays (indirect) taxes on his net purchases, and receives a lump-sum compensation  $s^i$ .

**Definition 11.1.** A compensated unemployment equilibrium is an allocation  $x^i$ ,  $i = 1, \dots, m$ , a rationing scheme  $l \in -\Omega$ , a tax vector  $t \in \Omega$ , a price vector  $p \in P$ , and a set of compensations  $s^i$ ,  $i = 1, \dots, m$ , such that

- a)  $x^i$  maximizes  $u^i$  on  $B^i(p, l, t, s^i)$ ,  $i = 1, \dots, m$ ,
- b)  $\sum_i (x^i - w^i) = 0$  and  $\sum_i t^T(x^i - w^i)^+ = \sum_i s^i$ ,
- c) for all  $j$ ,  $t_j > 0$  implies  $x^i_j - w^i_j > l_j$  for all  $i$ , and  $x^h_j - w^h_j = l_j$  for some  $h$  implies  $t_j = 0$ ,
- d) for all  $j$ ,  $p_j < \bar{p}_j$  implies  $t_j = 0$ , and  $p_j > \underline{p}_j$  implies  $l_j < x^i_j - w^i_j$  for all  $i$ ,
- e) for all  $i$ ,  $u^i(\tilde{x}^i) = u^i(x^i)$ , where  $\tilde{x}^i$  maximizes  $u^i$  in  $B^i(p, -w, t, 0)$ .

Condition b) states that all markets are in equilibrium and that the total tax revenue equals the total amount of compensations. Observe that if all markets are in equilibrium the latter follows from the fact that utility maximization implies that the optimal consumption is on the budget line and hence the value of the total excess demands plus the total tax revenue is equal to the total amount of compensations. Condition c) states that on each market not simultaneously supplies are constrained and a positive tax is levied on the net purchases. This complementarity between taxes and constraints implies that there is just one instrument on each market to equate demand and supply. Condition d) guarantees that there is no tax levied if the price is not on its upper bound and that there is no supply rationing if the price is not on its lower bound. Observe that both conditions correspond to the conditions c) and d) of a constrained equilibrium. Condition e) determines the unemployment compensation. The compensation for consumer  $i$  is such that the loss of utility due to the supply constraints is just compensated by the subsidy  $s^i$ , i.e., the optimal consumption in the unconstrained budget set with zero compensation is equally preferred by agent  $i$  to the optimal consumption in the constrained budget set with income compensation  $s^i$  8).

8) Observe that preferences are defined on the consumption set  $X^i$ , i.e., utility only depends on consumption, including leisure. In case utility also depends on whether or not having a job, the existence of an



In order to prove the existence of a compensated unemployment equilibrium, let  $T \in \Omega$  be a vector of sufficiently high taxes, i.e., nobody wants to buy anything of commodity  $j$  if  $T_j$  is the tax on net purchases of commodity  $j$  and there exists a commodity  $h$  with tax  $t_h$  equal to zero. The assumptions  $A_2$  and  $A_3$  of Section 2 guarantee the existence of such a vector  $T$  as long as the subsidies  $s^i$  are bounded. It can be easily observed that because of the monotonicity of the utilities the best element  $\tilde{x}^i$  in  $B^i(p, -w, t, 0)$  has a lower utility than the best element  $y^i$  in  $B^i(p, l, t, p^T w^i)$  for any  $l \leq 0$ . Hence, the monotonicity of the utilities guarantees that there exists an  $s^i \leq p^T w^i$  such that  $s^i$  just compensates for the supply rationing  $l$ . So, the compensations are bounded by  $\bar{p}^T w^i$ . For all  $q \in Q$  (see (7.1)) we set  $p(q)$  and  $l(q)$  as defined in (7.2) and (7.3), and define  $t_j(q)$  by

$$t_j(q) = \max(0, (q_j - \bar{p}_j) / \bar{p}_j) T_j, \quad j = 0, \dots, n.$$

Clearly  $t_j(q) = 0$  if  $q_j \leq \bar{p}_j$  and  $t_j(q) = -w_j$  if  $q_j \geq \bar{p}_j$ . Let  $x^i(q)$  be the optimal consumption in the budget set  $B^i(q) = B^i(p(q), l(q), t(q), s^i(q))$  with  $s^i(q)$  defined such that the utility of the optimal consumption  $\tilde{x}^i(q)$  in the budget set  $B^i(p(q), -w, t(q), 0)$  is equal to  $u^i(x^i(q))$ . From the monotonicity of the preferences it follows that  $s^i(q)$  is unique (and bounded by  $\bar{p}^T w^i$ ). From the strict quasi-concavity of the utility functions it follows that the excess demand function  $z: Q \rightarrow \mathbb{R}^{n+1}$  defined by  $z(q) = \sum_i (x^i(q) - w^i)$  is continuous. It can be proved that  $z$  has a zero point  $q^*$  in  $Q$  such that there exists a  $j$  with  $q_j^* \leq \bar{p}_j$  and an  $h \neq j$  with  $q_h^* \geq \bar{p}_h$ <sup>9)</sup>. Clearly, such a zero point yields an equilibrium  $x^i(q^*)$ ,  $s^i(q^*)$ ,  $i=1, \dots, m$ ,  $p(q^*)$ ,  $l(q^*)$ , and  $t(q^*)$ , with  $p_h(q^*) = \bar{p}_h$ ,  $l_h(q^*) = -w_h$ , and  $t_j(q^*) = 0$ . This gives us the next theorem.

**Theorem 11.2.** There exists a compensated unemployment equilibrium with  $t_j = 0$  for at least one  $j$ , and  $p_h = \bar{p}_h$  for at least one  $h \neq j$ .

Recall that in case of supply constrained equilibria there are two trivial zero points of  $z$  in  $Q$ , namely  $q = 0$  and  $q = 2\bar{p}$  (see Section 7). Also here the latter point yields a trivial equilibrium with  $t_j = T_j$  for all  $j$ , since for all  $i$ ,  $x^i = w^i$  is a maximal element in the budget set  $B^i(\bar{p}, -w, T, 0)$ . At  $q = 0$  however, we have  $t_j = 0$  for all  $j$ ,  $p = \bar{p}$  and  $l = 0$ . Because all taxes are equal to zero it follows from b) of Definition 11.1 that  $s^i$  must be equal to zero for all  $i$ . However,  $l = 0$  implies that  $s^i > 0$  unless  $x^i = w^i$  is maximal in the budget set  $\{x \in X^i \mid \bar{p}^T x \leq \bar{p}^T w^i\}$ . Hence,  $q = 0$

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unemployment compensation satisfying condition e) can not be guaranteed. That means, the negative impact of being unemployed on social life can not be compensated.

<sup>9)</sup> The existence of such a zero point can be proved by simplicial approximation theory and is beyond the scope of this paper.

does not yield a trivial equilibrium. Since the theorem says that there is an equilibrium with  $t_j = 0$  for at least one  $j$ , this implies the existence of a nontrivial equilibrium.

From the discussion above it follows that it is possible to compensate unemployed people for the loss of utility caused by the supply constraints and that the unemployment doles can be financed through taxes on the net purchases. Let us consider an example of an economy with two commodities and two types of agents. Agents of type A have initial endowments of commodity 1 and (almost) no endowments of commodity 2, while for agents of type B the opposite holds. So, at all prices, agents of type A supply commodity 1 and demand commodity 2 and reversely for agents of type B. Now take  $p_1 = 1$ , let  $p_2^*$  be the unique Walrasian equilibrium price for commodity 2, and suppose that  $p_2 < p_2^*$  is a fixed price for commodity 2. Since the price of commodity 2 is too low compared to the equilibrium price, we have that at this price system there is an excess supply of commodity 1 and an excess demand of commodity 2. So, a compensated equilibrium will be reached through rationing on the supplies of commodity 1 and levying a tax on the purchases of commodity 2. Consequently, agents of type B are not constrained and do not pay taxes. On the other hand agents of type A pay a tax on the purchase of commodity 2, while at least some of them are rationed in the supplies. In case there is only one agent of type A, this agent pays his own unemployment compensation. This is illustrated in the Edgeworth box of Figure 11.1. In this figure agent A faces a rationing  $t_1^A$  on his supply of commodity 1 and has to pay a tax  $t_2$  on the purchases of commodity 2. Moreover, consumer A receives a compensation  $s^A$ . Through this rationing, tax, and compensation system agent's A unconstrained optimal consumption  $x^A(p)$  under the budget restriction  $p^T x^A \leq p^T w^A$ , moves to the point  $x^B(p)$  and agent's A consumption becomes  $\bar{x}^A = w - x^B(p)$ . For consumer A this point yields the same utility as the unconstrained uncompensated optimal consumption  $\bar{x}^A$  under the budget restriction  $p^T x^A + t_2(x_2^A - w_2^A) = p^T w^A$ . Clearly, we have that in this point  $t_2(\bar{x}_2^A - w_2^A) = s^A$ , so that consumer A pays his own compensation. This outcome seems to be rather unsatisfactory. However, observe that this agent is not totally unemployed, because he can offer for sale an amount equal to the demand of the agent of type B. In case of non-uniform rationing with each agent either unrationed or (totally) unemployed the employed agents of type A pay taxes to finance the unemployment compensation of the others. If there is uncertainty about which agents of type A are unemployed, we can say that the tax paid by an agent of type A is a social insurance premium against the loss of income when losing his job, i.e., for when being excluded from the possibility to sell commodity 2.

Insert Figure 11.1.

The vector of sales taxes plays two roles in the model of a compensated unemployment equilibrium. Firstly, the unemployment compensation is financed from the tax revenue, as has been discussed above. Secondly, through the system of differentiated taxes the tax vector serves as an instrument to equalize demand and supply. Supply rationing occurs on markets of commodities with relative high prices, whereas there is a purchase tax on the commodities with relative low prices. In fact, for the latter commodities the demand rationing has been replaced by taxing the net purchases. Since we introduced the tax vector in order to finance the unemployment compensations the second role of the tax vector is figurative. Therefore, let us consider a model in which the unemployment compensations are financed through a tax on the income of the agents. In that case a percentage of the income is paid as an insurance premium against unemployment. Each agent faces a tax rate  $\alpha \in [0,1]$  to be paid over his tax income. For some positive vector  $a \in \Omega$ , the tax income of agent  $i$ ,  $i = 1, \dots, m$ , is defined by

$$I^i(x) = p^T(w^i - x - a)^+, \quad x \in X^i,$$

i.e., tax must be paid over the value of the total net sales of the commodities, except that for each commodity  $j$  there is some positive amount  $a_j$  which can be sold free of tax. Given a price  $p \in P$ , a rationing scheme  $l \leq 0$ , an income tax rate  $\alpha \in [0,1]$ , and a compensation  $s^i \in R_+$ , the budget set of consumer  $i$  becomes

$$B^i(p, l, \alpha, s^i) = \{x \in X^i \mid p^T x + \alpha I^i(x) \leq p^T w^i + s^i, \quad l \leq x - w^i\}.$$

**Definition 11.3.** An income tax unemployment equilibrium is an allocation  $x^i$ ,  $i = 1, \dots, m$ , a rationing scheme  $l \in -\Omega$ , an income tax rate  $\alpha \in [0,1]$ , a price vector  $p \in P$ , and a set of compensations  $s^i$ ,  $i = 1, \dots, m$ , such that

- $x^i$  maximizes  $u^i$  on  $B^i(p, l, \alpha, s^i)$
- $\sum_i (x^i - w^i) = 0$  and  $\sum_i s^i = \alpha \sum_i I^i(x^i)$ .
- for all  $j$ ,  $p_j > p_j$  implies  $l_j < x_j^i - w_j^i$  for all  $i$ ,
- for all  $i$ ,  $u^i(\bar{x}^i) = u^i(x^i)$  where  $\bar{x}^i$  maximizes  $u^i$  on the budget set  $B^i(p, -w, \alpha, 0)$  and  $\bar{x}^i$  on  $B^i(p, l - a, \alpha, s^i)$ .

Again, condition b) implies that total tax revenue equals total amount of compensations. The compensation of agent  $i$  is determined by d). The unemployment dole compensates the loss of utility due to the rationing  $l - a$ . So, if agent  $i$  wants to sell more than  $-l_j$ , but not more than  $-l_j + a_j$ , then he is not compensated for the



constraint on commodity  $j$ . This reflects the fact that for all  $j$ , a part  $p_j a_j$  of the value of the net sale is free of tax, i.e., unemployment is not insured as long as the unemployment is less or equal to some amount. The introduction of a (relatively small) tax free vector is not only reasonable, since small risks are typically not insured, but has also a technical reason. Suppose that  $a = 0$ . Then  $\alpha = 1$ , in which case all income is taxed away, implies that  $x^i = w^i$  is a maximal element in the budget set for all  $l \leq 0$  and  $s^i = 0$ . Hence, if  $a = 0$  for all  $l \leq 0$  we have a trivial equilibrium. For a positive vector  $a$  we still have that  $l = 0$  implies that  $x^i = w^i$  is a maximal element of the budget set and therefore yields a trivial equilibrium, because  $\alpha = 1$  implies that no agent wants to sell more than  $a_j$  of commodity  $j$  and hence  $s^i = 0$ . However  $l \neq 0$  may result in a positive supply of some commodity.

We have seen that in the model with taxes on net purchases there are  $n+1$  instruments to equilibrate the  $n+2$  equations of condition b). Now, however, there are  $n+2$  instruments, namely a rationing instrument on each market and the tax rate  $\alpha$ . However, by Walras' law we know that all equations hold as soon as  $n+1$  are equalized. It follows that an additional condition can be stated. The next theorem says that there exists an equilibrium with at least one unrationed commodity.

**Theorem 11.4.** There exists an income tax unemployment equilibrium with  $l_j = -w_j$  for at least one  $j$ .

For the proof, see van der Laan (1981).

We finally consider the case that  $p_j = \bar{p}_j = \hat{p}_j$  for all  $j$  and make some remarks under several assumptions which we will not further discuss. First, let us assume that there is a unique equilibrium with  $l_j = -w_j$  for some  $j$ . Let  $\hat{\alpha} = \alpha(\hat{p})$  be the tax rate at this equilibrium. Then it can be proved (see van der Laan (1981)) that for all  $\alpha \in [\hat{\alpha}, 1]$  there is an income tax unemployment equilibrium, with  $l_j = l_j(\alpha) \geq -w_j$ . We have already seen that for  $\alpha = 1$ ,  $l = 0$  yields a trivial equilibrium. In fact, when raising  $\alpha$  from  $\hat{\alpha}$  to 1 the rationing is tightened (not necessarily monotonically) from  $l_j = -w_j$  for at least one  $j$  to  $l_j = 0$  for all  $j$ . In other words, raising the tax rate decreases the employment possibilities. Further, assume that there is a unique Walrasian price system  $p^* = (p_0^*, \dots, p_n^*)^T$ . Then, if  $\hat{p} = p^*$  is the vector of fixed prices, the unique Walrasian equilibrium allocation is induced by the income tax unemployment equilibrium with  $l = -w$ ,  $\alpha = 0$ , and  $s^i = 0$  for all  $i$ . Moreover, for each  $\alpha \in (0, 1]$ , there exists an income tax unemployment equilibrium with  $l = l(\alpha) \geq -w$ . From this we come to the conclusion that a positive income tax rate at the Walrasian price system  $p^*$  induces unemployment. When we assume that  $\alpha(\hat{p})$  is continuous in  $\hat{p}$ , then for any given  $\alpha > 0$ , there is an income tax



unemployment equilibrium for all  $\hat{p}$  close enough to  $p^*$ . So, a positive tax rate stabilizes the system in the sense that for small (temporary) disturbances in the prices there still exists an equilibrium with the same income tax rate, i.e., the unemployment induced by  $\alpha$  absorbs disturbances of the equilibrium prices, as long as these disturbances are not too large.

## 12. Money adjustment

In the previous section we have seen that in a compensated unemployment equilibrium it is not possible to guarantee that there is neither rationing nor taxing on an a priori chosen numeraire commodity. However, in a monetary economy rationing or taxing on money does not seem to be a very satisfactory result. Therefore we discuss a modification of the compensated unemployment equilibrium, such that there is neither rationing nor taxing on the numeraire commodity money. In Section 9 we have seen that rationing on money in a supply-constrained equilibrium can be excluded by flexible money prices. In fact, rationing of money can be excluded by choosing the price level high enough. In an economy with rationing and taxing, however, we can deal with non-flexible money prices by allowing for the fact that the total amount of tax revenue and the total amount of lump-sum unemployment compensations may differ.

With the commodity indexed by  $j = 0$  as the numeraire commodity the set of admissible prices becomes

$$P = \{p \in \Omega \mid p_0 = 1, p_j \leq \bar{p}_j \text{ for all } j \neq 0\}.$$

with for all  $j \neq 0$ ,  $0 < p_j \leq \bar{p}_j < \infty$ .

**Definition 12.1.** A monetary compensated unemployment equilibrium is an allocation  $x^i$ ,  $i = 1, \dots, m$ , a rationing scheme  $l \in -\Omega$ , a tax vector  $t \in \Omega$ , a price vector  $p \in P$ , and a set of lump-sum compensations  $s^i$ ,  $i = 1, \dots, m$ , such that

- 1) the conditions a), c), d), and e) of Definition 11.1 hold,
- 2)  $t_0 = 0$ , and  $l_0 = -w_0$ ,
- 3)  $\sum_i (x^i_j - w^i_j) = 0$  for  $j = 1, \dots, n$ , and  $\sum_i x^i_0 - \sum_i w^i_0 = \sum_i s^i - \sum_i t^T (x^i - w^i)^+$ .

Condition 2) means that neither taxing nor rationing on money is allowed. This condition also excludes the trivial equilibrium with  $t_j = T_j$ ,  $j = 1, \dots, n$ , with  $T_j$  large enough. The first part of condition 3) requires that all non-money markets are in equilibrium. The second part follows then immediately from the fact that utility maximization implies that the total value of the total excess demands must be equal

to the total amount of compensations minus the total tax revenue. It says that the total difference between the terminal and initial holdings of money equals the difference between the total amount of lump-sum subsidies and the total tax revenue.

Theorem 12.2. There exists a monetary compensated unemployment equilibrium.

For the proof we refer to van der Laan (1980, Theorem 4).

Clearly, a monetary compensated unemployment equilibrium does not imply that the total tax revenue  $\sum_i t^T(x^i - w^i)^+$  equals the total lump-sum compensations  $\sum_i s^i$ . In Section 11 the only role of the government was to collect the taxes and to distribute the revenue. However, in a monetary economy the government must also increase or decrease the total amount of money in order to obtain equilibrium. This results in respectively inflationary or deflationary impulses.

In the two sector model of Malinvaud the government demand for the consumption good serves as an instrument for economic policy. By choosing an exogenous demand the government has the possibility to increase or to decrease the demand for the consumption good. The demand of the government results in an increase or decrease of the total amount of money equal to the value of the demand minus the income of the government. In the Malinvaud model this income comes from taxing away the profits of the producers. So, the real savings of the consumers are equal to the money creation or destruction by the government, being the difference between the spending of the government on the consumption good and the income of the government. Dehez and Gabszewicz (1977, 1979) presented a Malinvaud-type model in which the government appears as an active economic agent reducing excess demand or absorbing excess supply on the commodity market. In a dynamic setting, this behaviour forces the economy towards a stationary state, i.e., an equilibrium such that the total initial amount of money equals the total terminal holdings of money and hence real savings are equal to zero. More precisely, assume that in each period the initial holding of money of any consumer is equal to his terminal holding of money at the end of the previous period, with the initial holdings at the first period given, then Dehez and Gabszewicz showed that under some assumptions the sequence of disequilibrium states for the subsequent periods converges to a stationary equilibrium state.

In the model of a monetary unemployment equilibrium we have that the money creation or destruction is endogeneously determined by the difference between the unemployment compensations and the tax revenue. Let us consider the consequences of this policy for the Malinvaud-type example of an exchange economy

as given in Section 5 (see Figure 5.1). Let  $p$  be a vector of fixed prices in region I, i.e., the region with excess supply on both markets. In our model this yields an equilibrium with supply rationing on both markets and hence unemployment compensations without tax revenue. So, the amount of money increases and the terminal holdings of money will be higher than the initial holdings. Hence, in the next period the initial holding of money will be higher. Assuming that for all non-money commodities the initial endowments are constant over the periods, the Walrasian prices will move upwards because of the homogeneity of the equilibrium prices and the money endowments. This is illustrated in Figure 12.1, where the Walrasian prices are assumed to move upwards from  $(p^W_1, p^W_2)$  along the curve  $WW'$ . If the fixed price vector  $p$  is on this curve, the sequence of Walrasian equilibrium prices will converge to  $p$ . On the other hand, the Walrasian prices will move downwards along  $WW'$  if  $p$  is initially in region III, i.e., the region with excess demand on both markets. Consequently, the model results into an equilibrium with taxes on the demands of both commodities and hence we have positive tax revenues without unemployment compensations. Again, the sequence of Walrasian equilibrium prices will converge to the vector  $p$  of fixed prices if  $p$  is on the curve  $WW'$ . As soon as the Walrasian prices become equal to  $p$  a stationary state has been reached. So, we have that the model is similar to the model of Dehez and Gabszewicz, except that the instrument of varying the demand of the consumption goods by the government has been replaced by the endogeneously determined system of taxes and unemployment compensations.

Insert Figure 12.1.

In case the vector of fixed prices does not lie on the curve  $WW'$ , then the Walrasian prices move along the curve  $WW'$  until a price vector  $p^*$  has been reached such that the real savings induced by the price system  $p$  are equal to zero. This is illustrated in Figure 12.2, in which the broken lines divide the price spaces into regions I, II, III, and IV according to the final Walrasian price system  $p^*$ . Observe that according to  $p^*$  the system of fixed prices lies in region II, while  $p$  lies in region I according to the initial Walrasian price system  $p^W$ . So, the regime of excess supplies on both markets switches to a mixed regime of excess demand on one market and excess supply on the other market when as a consequence of the real savings the Walrasian prices move from  $p^W$  to  $p^*$ .

Insert Figure 12.2.



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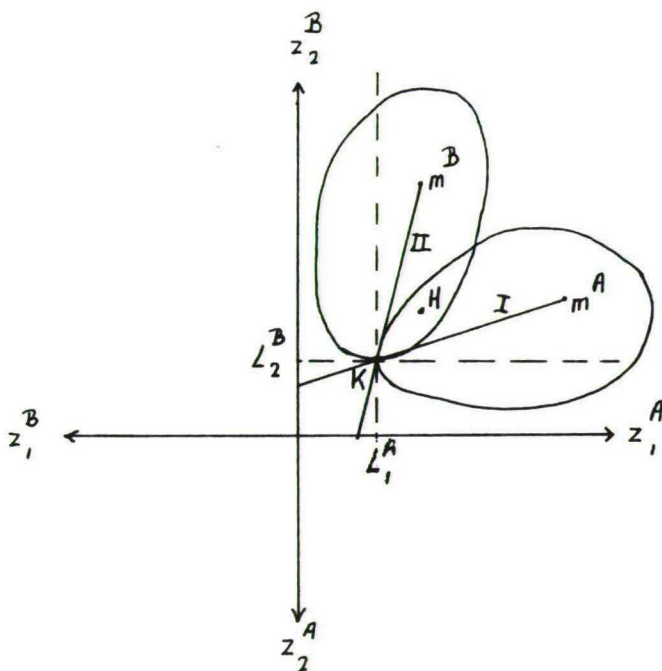


Figure 4.1.

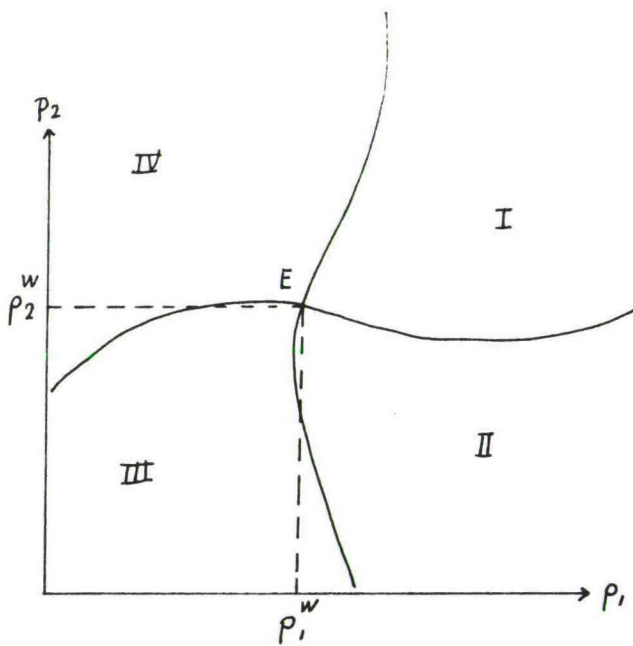


Figure 5.1.

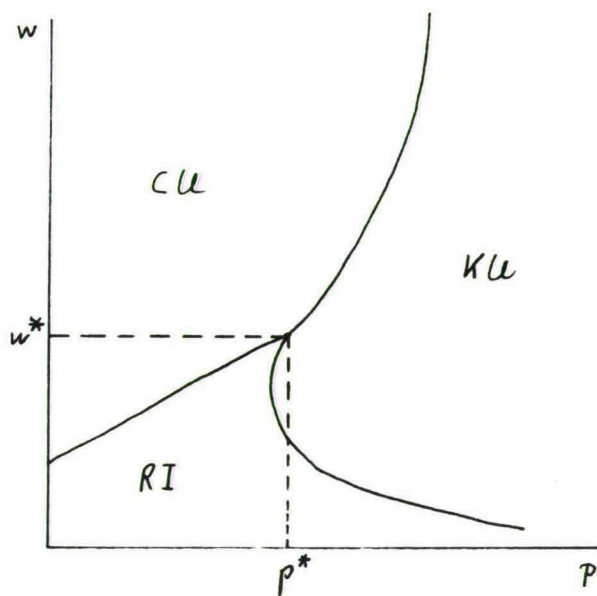


Figure 5.2.



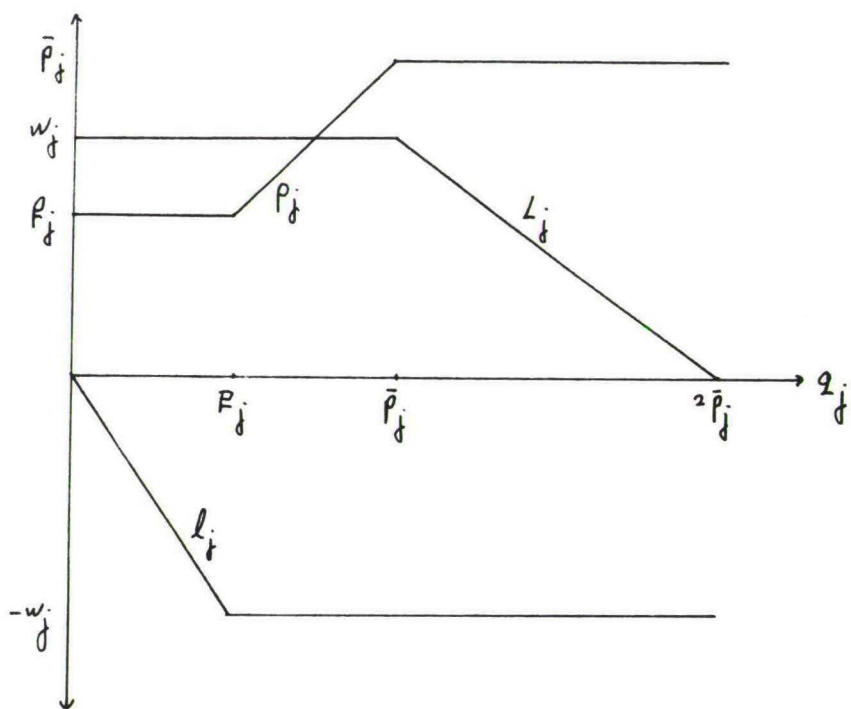


Figure 7.1.

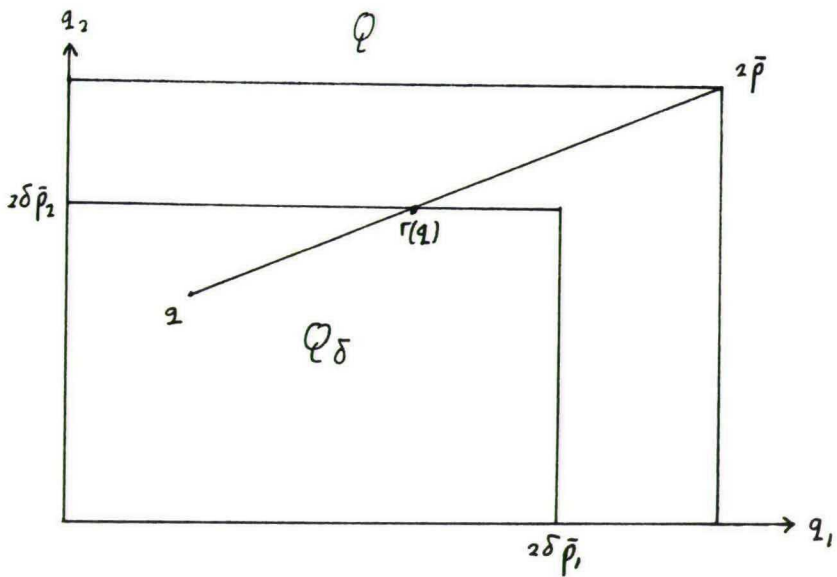


Figure 7.2.

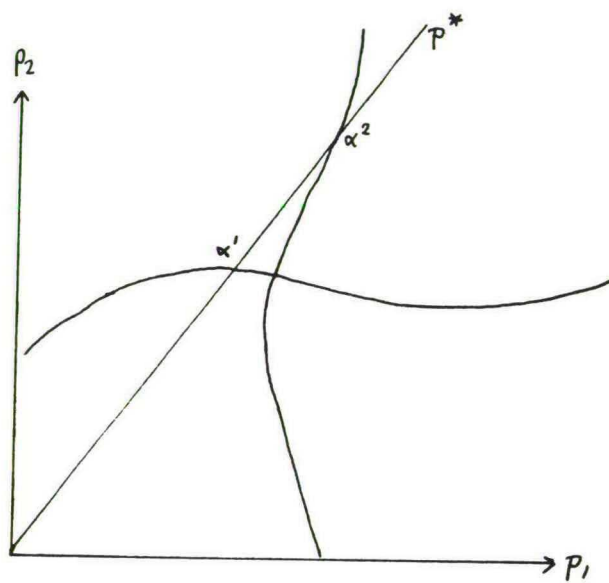


Figure 10.1.

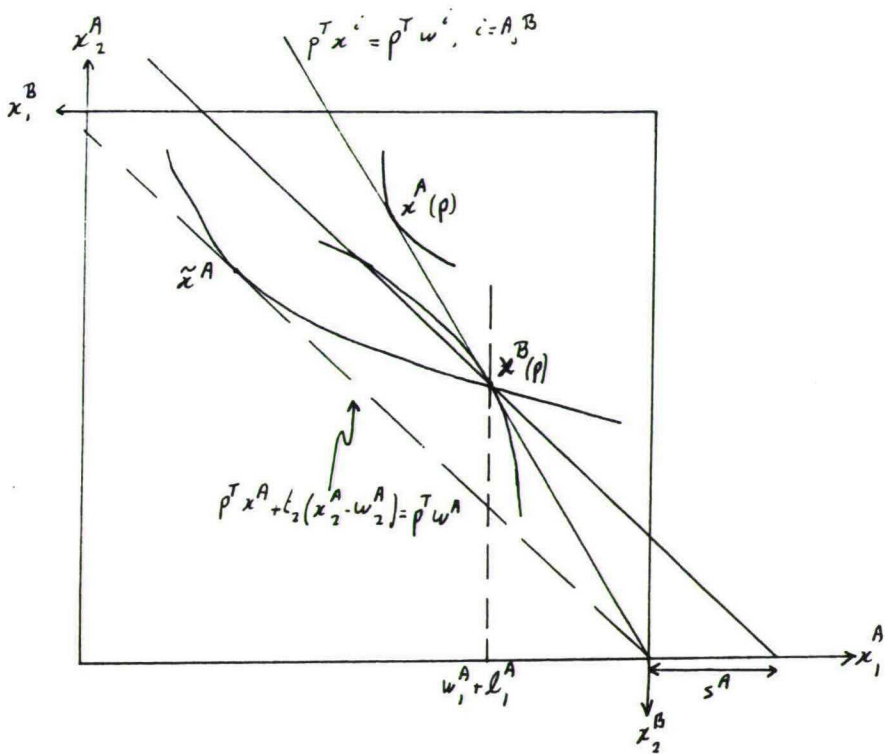


Figure 11.1.



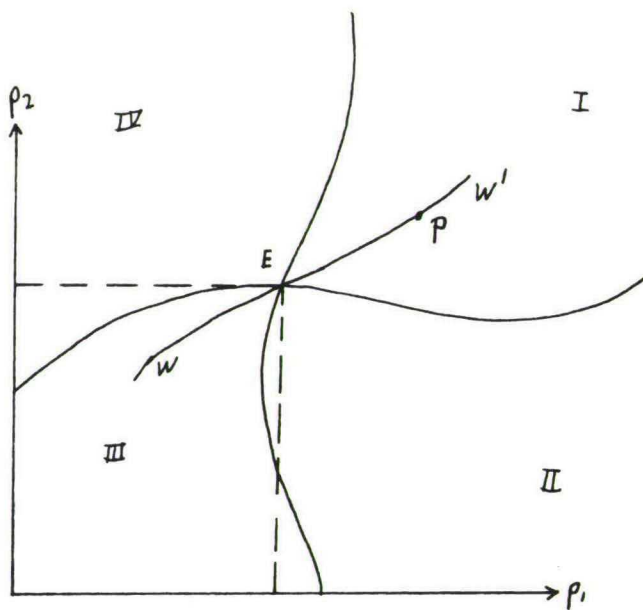


Figure 12.1.

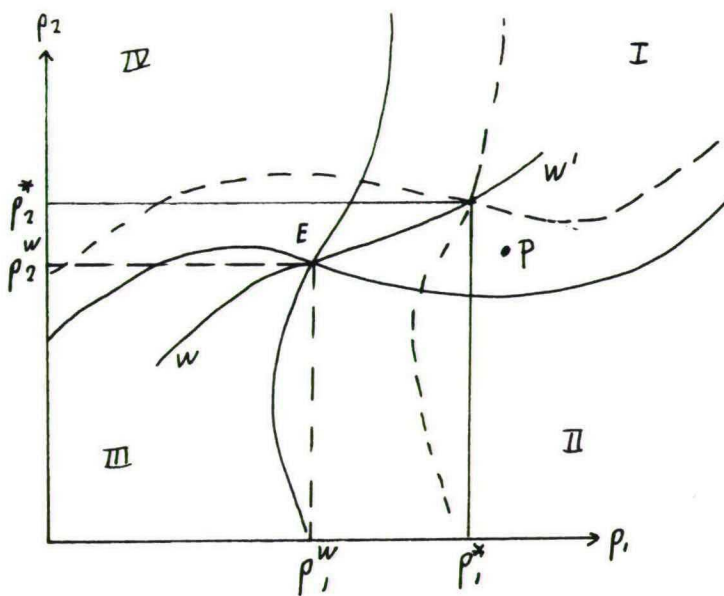


Figure 12.2.

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